Sede amministrativa: Università degli studi di Padova
Dipartimento di Matematica

DOTTORATO DI RICERCA IN MATEMATICA
INDIRIZZO MATEMATICA
CICLO XXVIII

SOME RESULTS IN NONLINEAR PDES: LARGE DEVIATION PROBLEMS, NONLOCAL OPERATORS, AND STABILITY FOR SOME ISOPERIMETRIC PROBLEMS

Direttore della scuola: Ch.mo Prof Pierpaolo Soravia
Coordinatore di indirizzo: Ch.mo Prof Franco Cardin
Supervisore: Ch.mo Prof Martino Bardi

Dottoranda: Daria Ghilli

Anno accademico 2014/2015

Ai miei genitori

Acknowledgements

I wish to express my deep gratitute to my supervisor, prof. M. Bardi, for having followed my research formation during my PhD and having driven me into these subjects of research. I sincerely thank him for having always played a stimulating role, for his advices and his precious guide.

I would like to thank sincerely Annalisa Cesaroni, who strongly contributed to my mathematical formation both by her mathematical advices and also by her personal support given to me in these three years.

I thank Paolo Salani, who followed me during my very first research experience in my master thesis at the University of Florence and he kept on supporting me also after.

I want to devote my special gratitude to prof. Guy Barles, who has taken me under his supervision during my research period at the Université François Rabelais de Tours.

I sincerely thank him for his constant support, for the several discussions on maths we had the chance to do, for his advices, for his patience, for his good mood!, for his kindness. He has been very important to my formation and to my growth.

I also want to thank sincerely all the Laboratoire Mathématique et Physique Théorique de Tours for the warm welcome given to me. I wish to give a special thank to Emmanuel Chasseigne for his kindness and his availability each time I had questions. Un merci particulier à mon professeur de français Xavier, et à mes amies Gabriel, Thao et Shouman.

I sincerely thank all my PhD collegues of the University of Padua for these three years together, Daniele, Davide, Buoso, Franceschino, Gabo e Maria, Gigi, Laura, Luisa, Marta, Miche, Vale and I thank Fede, for moderncv, for parindent, for the blowup function, for the pizza on sundays, for his support.

I thank all the people and professors I had the chance to know, to discuss with and to listen during these three years. In particular I thank prof. Hasnaa Zidani who contributed to the continuation of my research activity in these last months and the referee of my thesis for the interesting advices which contributed to the improvement of the thesis.

And finally, I want to thank my friends Ale, Andre, Cala, Fra, Leti, Lollo, Lore, Lu, Mari, Marghe, Nicco, Pierpa, Risto, Simo, Vero, and I thank my family and my parents, for being always with me.

Abstract

This thesis is concerned with various problems arising in the study of nonlinear elliptic PDE. It is divided into three parts.

In the first part we consider the short time behaviour of stochastic systems affected by a stochastic volatility evolving at a faster time scale. Our mathematical framework is that of multiple time scale systems and singular perturbations. We are concerned with the asymptotic behaviour of a logarithmic functional of the process, which we study by methods of the theory of homogenization and singular perturbations for fully nonlinear PDEs. We point out three regimes depending on how fast the volatility oscillates relative to the horizon length. We provide some financial applications, namely we prove a large deviation principle for each regime and apply it to the asymptotics of option prices near maturity.

In the second part we are concerned with the well-posedness of Neumann boundary value problems for nonlocal Hamilton-Jacobi equations related to jump processes in general (enough smooth) domains. We consider a nonlocal diffusive term of censored type of order less than 1 and Hamiltonian both in coercive form and in noncoercive Bellman form, whose growth in the gradient make them the leading term in the equation. We prove a comparison principle for bounded sub-and supersolutions in the context of viscosity solutions with generalized boundary conditions, and consequently by Perron's method we get the existence and uniqueness of continuous solutions. We give some applications in the evolutive setting, proving the large time behaviour of the associated evolutive problem under suitable assumptions on the data.

In the last part we present some stability results for a class of integral inequalities, the Borell-Brascamp-Lieb inequality and we strengthen, in two different ways, these inequalities in the class of power concave functions. Then we present some applications to prove analogous quantitative results for certain type of isoperimetric inequalities satisfied by a wide class of variational functionals that can be written in terms of the solution of a suitable elliptic boundary value problem. As a toy model, we consider the torsional rigidity and prove quantitative results for its Brunn-Minkowski inequality and for its consequent (Urysohn type) isoperimetric inequality.

Questa tesi si occupa di vari problemi che sorgono nello studio di equazioni alle derivate parziali ellittiche e paraboliche. La tesi è divisa in tre parti.

Nella prima parte studiamo il comportamento per tempi brevi di sistemi dinamici a volatilità stocastica che evolve in una scala temporale più veloce. Ci occupiamo di perturbazioni singolari di sistemi a scala temporale multipla. Il nostro primo obiettivo è lo studio del comportamento asintotico di un funzionale logaritmico del processo stocastico, attraverso i metodi della teoria dell' omogeneizzazione e delle perturbazioni singolari per equazioni alle derivate parziali completamente non lineari. Individuiamo tre regimi a seconda della velocità con cui la volatilità oscilla rispetto alla lunghezza dell'orizzonte temporale. Inoltre forniamo alcune applicazioni finanziarie, in particolare proviamo un principio di grandi deviazioni in ogni regime e lo applichiamo per derivare una stima asintotica dei prezzi di opzioni vicino alla maturità e una formula asintotica per la volatilità di Black-Scholes implicita.

Nella seconda parte studiamo la buona definizione di problemi al contorno di tipo Neumann, in domini generali (sufficientemente regolari), per equazioni tipo Hamilton-Jacobi con termini non locali che derivano da processi discontinui a salti. Consideriamo un termine diffusivo non locale di tipo "censored", di ordine strettamente minore di 1, e un' Hamiltoniana, sia in forma coerciva sia di tipo Bellman non necessariamente coerciva, la cui crescita nel gradiente la rende il termine principale nell'equazione. Dimostriamo un principio di confronto per sotto e sopra soluzioni limitate (in senso di viscosità) con condizioni al contorno generalizzate, e di conseguenza tramite il metodo di Perron otteniamo l'esistenza e l'unicità di soluzioni continue. Diamo alcune applicazioni nel caso evolutivo, dimostrando la convergenza per tempi grandi della soluzione del problema evolutivo alla soluzione del problema stazionario associato, supponendo opportune ipotesi sui dati.

Nell'ultima parte presentiamo alcuni risultati di stabilità per una classe di diseguaglianze integrali, le disuguaglianze Borrell-Brascamp-Lieb e rafforziamo, in due modi diversi, queste disuguaglianze nella classe di funzioni a potenza concava. Come applicazione di questo risultato, presentiamo analoghi risultati quantitativi per alcuni tipi di disuguaglianze isoperimetriche soddisfatte da un'ampia classe di funzionali variazionali che possono essere scritti in termini della soluzione di un opportuno problema al contorno ellittico. Come modello giocattolo, consideriamo la rigidità torsionale e dimostriamo risultati quantitativi per la sua disuguaglianza Brunn-Minkowski e per la sua conseguente disuguaglianza isoperimetrica di tipo Urysohn.

Contents

Basic notation

$\mathbb{R}^+$	The set of positive real numbers.
$\mathbb{R}^d$	The d dimensional euclidean space, $d \geq 1$.
$x \cdot y$ (also (x,y))	The scalar product $\sum_{i=1}^d x_i y_i$ of two vectors $x = (x_1, \cdots, x_d)$ and $y = (y_1, \cdots, y_d)$.
$B_r(x), B_r$	The open ball of radius r centered at x. If x is omitted, we assume $x = 0$.
S^{d-1}	The unitary sphere in $\mathbb{R}^d$.
Ω	will be a domain of $\mathbb{R}^d$.
$C(\Omega)$	The space of continuous functions on Ω.
$BC(\Omega)$	The space of bounded and continuous functions on Ω.
$BUSC(\Omega)$	The space of bounded upper semicontinuous functions on Ω.
$BLSC(\Omega)$	The space of bounded lower semicontinuous functions on Ω.
$C^k(\Omega)$	The space of continuous functions on Ω with continuous derivatives of order j, $j = 1, \cdots, k$.
$C^{2,\gamma}(\Omega)$	The space of functions in $C^2(\Omega)$, with γ-holder continuous second derivatives.
$\frac{\partial f}{\partial x_i}, Df$	Partial derivatives with respect to the i-th variable and gradient vector of f.
$\frac{\partial^2 f}{\partial_{x_j x_i}}, D^2 f$	Partial derivatives with respect to the j-th variable of $\frac{\partial f}{\partial x_i}$ and Hessian matrix of f.
$n = n(x)$	The outer normal vector at a point $x \in \partial\Omega$.
$\frac{\partial f}{\partial n}$	Normal derivative $n \cdot Df$.
$\|\|f\|\|_\infty$	The supremum norm $\sup_{x\in\Omega} \|f(x)\|$ of a function $f : \Omega \to \mathbb{R}$.
$\|\|f\|\|_{L^\infty(\Omega)}$	$\inf\{C \geq 0 : \|f(x)\| \leq C$ almost everywhere in$\Omega\}$.
$L^\infty(\Omega)$	The space of functions $f : \Omega \to \mathbb{R}$ such that $\|\|f\|\|_{L^\infty(\Omega)} < +\infty$
$W^{2,\infty}(\Omega)$	The space of functions in $L^\infty(\Omega)$ with first and second weak derivative in $L^\infty(\Omega)$.
$\mathbf{M}^{r,d}$	The space of matrices $r \times d$.
$\|\|A\|\|_\infty, A \in \mathbf{M}^{r,d}$	$\max_i \sum_j \|A_{ij}\|$

Introduction

This thesis is divided into three part. The first part deals with small time behaviour of stochastic systems affected by a stochastic volatility evolving at a faster time scale by viscosity methods. In the second part we study existence and uniqueness of Neumann boundary value problems for nonlocal equations related to discontinuous jump processes. In the third part we present quantitative stability results for a certain class of isoperimetric inequalities. In the first two parts we use methods taken mainly from the theory of viscosity solutions. The third part is different for both contents and methods, which are taken mainly from convex analysis and geometry.

Part I-Large deviations of some fast stochastic volatility models by viscosity methods

In Part I we present the results of [17] and [18] carried out in collaboration with Martino Bardi and Annalisa Cesaroni.

We are interested in stochastic differential equations with two small parameters $\varepsilon > 0$ and $\delta > 0$ of the form

$$\begin{cases} dX_t = \varepsilon\phi(X_t,Y_t)dt + \sqrt{2\varepsilon}\sigma(X_t,Y_t)dW_t & X_0 = x \in \mathbb{R}^n, \\ dY_t = \frac{\varepsilon}{\delta}b(Y_t)dt + \sqrt{\frac{2\varepsilon}{\delta}}\tau(Y_t)dW_t & Y_0 = y \in \mathbb{R}^m, \end{cases} \tag{1}$$

where W_t is a standard r-dimensional Brownian motion, and the matrix τ is non-degenerate.

This is a model of systems where the variables Y_t evolve at a much faster time scale $s = \frac{t}{\delta}$ than the other variables X_t. The second parameter ε is added in order to study the small time behavior of the system, in particular the time has been rescaled in (1) as $t \mapsto \varepsilon t$.

Passing to the limit as $\delta \to 0$, with ε fixed, is a classical singular perturbation problem. Its solution leads to the elimination of the state variable Y_t and the reduction of the dimension of the system from $n+m$ to n and to the definition of an averaged limit system defined in $\mathbb{R}^n$ only. Of course the limit problem keeps some information on the fast part of the system.

There is a large mathematical literature on singular perturbation problems, both in the deterministic ($\sigma \equiv 0, \tau \equiv 0$) and in the stochastic case. In particular we mention some mathematical contributions most related to our work and methods, such as the monographs [121], [115], the memoir [3] (see also the references therein).

Our aim is to study the asymptotic behavior of the system (1) as both the parameters go to 0 and we expect different limit behaviors depending on the rate ε/δ. Therefore we put

$$\delta = \varepsilon^{\alpha}, \text{ with } \alpha > 1.$$

We consider a functional of the trajectories of (1) of the form

$$v^{\varepsilon}(t,x,y) := \varepsilon \log E\left[e^{h(X_t)/\varepsilon} | (X.,Y.) \text{ satisfy (1)}\right], \tag{2}$$

where h is a bounded continuous function. We observe that the logarithmic form of this payoff is motivated by the applications to large deviations that we want to give.

It is known that v^{ε} solves the Cauchy problem with initial data $v^{\varepsilon}(0,x,y) = h(x)$ for a fully nonlinear parabolic equation in $n+m$ variables. Letting $\varepsilon \to 0$ in this PDE is a regular perturbation of a singular perturbation problem for an HJB equation, where the fast variable y lives in $\mathbb{R}^m$.

Our first aim is to prove that, under suitable assumptions, the functions $v^{\varepsilon}(t,x,y)$ converge to a function $v(t,x)$ characterised as the solution of the Cauchy problem for a first order Hamilton-Jacobi equation in n space dimensions

$$v_t - \bar{H}(x, Dv) = 0 \text{ in }]0,T[\times\mathbb{R}^n, \quad v(0,x) = h(x), \tag{3}$$

for a suitable effective Hamiltonian $\bar{H}$.

We observe that the existing techniques to treat this kind of problems have been developped so far mainly under assumptions implying some kind of boundedness of the fast variable. We refer mainly to the methods of [4], stemming from the pioneering paper of Lions-Papanicolau-Varadhan [128] and Evans' [76] on periodic homogenization and its extensions to singular perturbations [1–3].

A classical hypothesis is the periodicity with respect to Y_t of the coefficients of the stochastic system, which in particular implies the periodicity in y of the solutions v^{ε}. In Chapter 1 we carry out our analysis under this main assumption, treating what we call the *periodic case*. In the periodic case, the convergence is quite standard once the effective problem is identified and a comparison principle is proved. The most significant part in Chapter 1 is the identification of the *effective Hamiltonian* $\bar{H}$, which is obtained by solving a suitable cell problem. As usual in the theory of homogenization for fully nonlinear PDEs, this is an additive eigenvalue problem. It turns out to have different forms in the following three regimes depending on α:

$$\begin{cases} \alpha > 2 & \text{supercritical case,} \\ \alpha = 2 & \text{critical case,} \\ \alpha < 2 & \text{subcritical case.} \end{cases}$$

More precisely, in the supercritical case the cell problem involves a linear elliptic operator and $\bar{H}$ has the explicit formula

$$\bar{H}(x,p) = \int_{\mathbb{T}^m} |\sigma(x,y)^T p|^2 \, d\mu(y)$$

where μ is the invariant probability measure on the m-dimensional torus $\mathbb{T}^m$ of the stochastic process

$$dY_t = b(Y_t)dt + \sqrt{2}\tau(Y_t)dW_t.$$

In the critical case the cell problem is a fully nonlinear elliptic PDE and $\bar{H}$ can be represented in various ways based, e.g., on stochastic control. Finally, in the subcritical case the cell problem is of first order and nonlinear, and a representation formula for $\bar{H}$ can be given in terms of deterministic control. In particular, under the condition $\tau\sigma^T = 0$ of non-correlation among the components of the white noise acting on the slow and the fast variables in (1), we have

$$\bar{H}(x,p) = \max_{y \in \mathbb{R}^m} |\sigma^T(x,y)p|^2.$$

Let us mention that an important step of the method is the comparison principle for the limit Cauchy problem (3), ensuring that the weak convergence of the relaxed semilimits is indeed uniform, as well as the uniqueness of the limit.

In Chapter 2 we consider fast mean-reverting stochastic volatility systems as in (1) where the fast variable is unbounded and actually lives in $\mathbb{R}^m$, under suitable assumption on the coefficients of the system. The non compactness is replaced by some condition implying ergodicity, i.e that the process Y_t has a unique invariant distribution (the long-run distribution) and that in the long term it becomes independent of the initial distribution.

Following the line of Chapter 1, we consider a logarithmic functional of the trajectories of (1), and we prove that its limit as $\varepsilon \to 0$ can be characterised as the unique solution of a suitable limit problem.

In this part the main difficulties are due to the unboundedness of the process Y_t. Mainly because of this difficulty, we have to assume further hypotheses on the coefficients of the stochastic system; in particular we manage to treat processes mainly of Ornstein-Uhlenbeck type, that is

$$Y_t = (\mu - Y_t)dt + \tau(Y_t)dW_t$$

where $\mu \in \mathbb{R}^m$ is a vector, and τ is bounded and uniformly non degenerate. Note that the drift is dependent on the current value of the process in the following way (take for simplicity $m = 1$): if the current value of the process is less than the (long-term) mean, the drift will be positive; if the current value of the process is greater than the (long-term) mean, the drift will be negative. This gives the process the name "mean-reverting".

The motivation behind the analysis of such kind of systems relies in the fact that the assumption of periodicity of Chapter 1 seems a bit restrictive for the financial applications we have in mind, in particular it does not appear natural in order to model volatility in financial markets, according to the empirical data and the discussion presented in [86] and the references therein.

We remark that in this part we manage to treat the critical case ($\alpha = 2$) and the supercritical case ($\alpha > 2$), whereas we don't deal with the subcritical case ($\alpha < 2$). Indeed in the subcritical case the cell problem is not solvable in general; this is essentially due to the fact that the ergodicity of the fast process plays no role in the cell problem in this case.

The main issues are the resolution of the cell problem, the identification of the limit Hamiltonians and the convergence result. Our methods are based on the use of the approximate δ-cell problem. A key result is the global Lipschitz bound for the solution of the δ-approximate cell problem, uniform in δ, proved in Chapter 2, Proposition 2.5.5 (critical case) and Proposition 2.5.14 (supercritical case). The proof is inspired in some part by a method due to Ishii and Lions [112] (see also [67],[28] and the references therein), which essentially allows to take profit of the uniform ellipticity of the equation to control the Hamiltonian terms. However, we remark that usually the Ishii-Lions method allows to achieve bounds which depend on the L^∞-norm of the solution (at least if we do not assume any periodicity), whereas our aim is to establish a global estimate in all the space independent of such norm. The fundamental hypothesis which enables us to achieve our result consists in assuming that the fast processes we consider are mainly of Ornstein-Uhlenbeck type. Note also that we deal with both linear Hamiltonians in the gradient (in the supercritical case) and superlinear quadratic Hamiltonians (in the critical case).

In the critical case the proof is carried out in three steps. We prove first an uniform local Lipschitz bound for the solution of the δ-cell problem (see Chapter 2, Section 2.5, Lemma 2.5.1). The proof of the local bound is carried out by the Bernstein method relying on the coercivity in the gradient of the cell equation (which, in the critical case, is an uniformly elliptic second order equation with quadratic Hamiltonian in the gradient). Note that, thanks to this local gradient bound, we are able to consider fast processes which coincide with the Ornstein-Uhlenbeck process only outside some ball. Moreover, it allows us quite general assumptions on the stochastic volatility (see assumption (S), Chapter 2, Section 2.1.1) . As a second step we prove a global Hölder bound not uniform in δ (see Proposition 2.5.2) by the Ishii-Lions method, relying mainly on the uniform ellipticity and on the Ornstein-Uhlenbech nature of the process Y_t. Finally, we achieve the global uniform Lipschitz bound using the first two steps and again relying deeply on the fact that the process Y_t coincides with the Ornstein-Uhlenbeck process (outside a ball) and on assumption (S) on the volatility. We remark that the proof is non standard mainly because we do not use any compactness or periodicity on the coefficients and our result is independent of δ and holds in all the space.

On the contrary, in the supercritical case the cell problem is an uniformly elliptic equation linear in the gradient. Since in this case we are not able to prove an analogous local bound as in Lemma 2.5.1, we strengthen the assumption on the fast process Y_t and we consider the Ornstein-Uhlenbeck process in all the space. Note also that in the supercritical case there is no need of the assumption (S) on the volatility. For further remarks we refer to Section 2.5, subsection 2.5.2. Once we have that Y_t is Ornstein-Uhlenbeck in all the space (and since we do not need (S)), the proofs of the Hölder bound and of the global uniform Lipschitz bound are analogous and even easier than in the critical case.

Let us recall some results in the literature related to gradient bounds for similar kind of equations. Gradient bounds for superlinear-type Hamiltonians can be found in Lions [126] and Barles [20], see also Lions and Souganidis [129] and Barles and Souganidis [32]. Recently, Hölder bounds for nonlinear degenerate parabolic equations were proved in Cardaliaguet and Sylvestre [55]. However, we remark that, in the previous works the bound depends usually on the L^∞-norm of the solution (that is, on δ, when dealing with the δ-cell problem), whereas, on the contrary, our aim is to find a bound which is independent of such parameter. In [32] some results independent of the L^∞ norm of the solutions are established but in periodic environment. We recall also the result of [53] by Capuzzo-Dolcetta, Leoni, Porretta for coercive superlinear Hamiltonians, where a uniform gradient bound is established, but in some Hölder norm and only in bounded domains. We refer also to Barles [21], Cardaliaguet [54]. Recently, uniform Lipschitz bound on the torus for analogous equations as ours (and more general) has been established by Ley and Duc Nguyen in [125].

As already hinted above, the first issue is the identification of the limit Hamiltonian through the resolution of the cell problem which is now defined in all the space. The existence of a limit Hamiltonian and of the corrector is proved by the use of the approximate δ-cell problem. The main result which allows us to conclude the existence is the uniform gradient estimate for the solution of the δ-cell problem. For the uniqueness of the limit Hamiltonian, we proceed differently in the critical and supercritical case. In the critical case, we rely on the ergodicity of the process Y_t and on the results of Ichihara [107], where ergodic type Bellman equations are studied in the case of a nonlinear quadratic term. On the contrary, in the supercritical case, we rely on the results of Bardi, Cesaroni, Manca in [15], where the uniqueness of a limit Hamiltonian is proved (but note that no existence of the true corrector is proved in [15]).

The main result is the convergence of the functions v^ε to the solution of the limit problem. Our techniques are based on the perturbed test function method of [76], [4], with some relevant adaptations to the unbounded setting. In order to deal with the non compactness of the fast variable, we deeply rely on the ergodicity of the fast process, which is encoded in the existence of a *Lyapounov function* (see Chapter 2, Section 2.2). A key result used in the convergence is, again, the global gradient bound of the corrector (see Chapter 2, Proposition 2.5.6 and Proposition 2.5.15), which we use in the proof of the convergence mainly to deal with the difficulties coming from the nonlinearity in the gradient of our equation.

The main application of the convergence result is a large deviations analysis of (1). We prove that the measures associated to the process X_t in (1) satisfy a Large Deviation Principle (briefly, LDP) with good rate function

$$I(x;x_0,t) := \inf\left[\int_0^t \bar{L}\left(\xi(s),\xi'(s)\right)\,ds \,\middle|\, \xi \in AC(0,t),\ \xi(0)=x_0, \xi(t)=x\right],$$

where $\bar{L}$ is the *effective Lagrangian* associated to $\bar{H}$ via convex duality. In particular we get that

$$P(X_t^\varepsilon \in B) = e^{-\inf_{x\in B}\frac{I(x;x_0,t)}{\varepsilon}+o(\frac{1}{\varepsilon})},\ \text{ as } \varepsilon \to 0$$

for any open set $B \subseteq \mathbb{R}^n$. We prove the large deviation principle in Chapter 1 for the three different regimes of the periodic case. A similar result holds also for the systems considered in Chapter 2 for $\alpha \geq 2$.
Following [79], we also apply this result to derive an estimate of option prices near maturity and an asymptotic formula for the Black-Scholes implied volatility.

Our first motivation for the study of systems of the form (1) comes from financial models with stochastic volatility. In such models the vector X_t represents the log-prices of n assets (under a risk-neutral probability measure) whose volatility σ is affected by a process Y_t driven by another Brownian motion, which is often negatively correlated with the one driving the stock prices (this is the empirically observed leverage effect, i.e., asset prices tend to go down as volatility goes up).

An important extension of the stochastic volatility approach was introduced recently by Fouque, Papanicolaou, and Sircar in the book [86] (see, in particular, Chapter 3). The idea is trying to describe the bursty behavior of volatility: in empirical observations volatility often tends to fluctuate to a high level for a while, then to a low level for another small time period, then again at high level, and so on, for several times during the life of a derivative contract. These phenomena are also related to another feature of stochastic volatility, which is mean reversion. A mathematical framework which takes into account both bursting and mean reverting behavior of the volatility is that of multiple time scale systems and singular perturbations. In this setting volatility is modeled as a process which evolves on a faster time scale than the asset prices and which is ergodic, in the sense that it has a unique invariant distribution (the long-run distribution) and asymptotically decorellates (in the sense that it becomes independent of the initial distribution). We refer the reader to the book [86] and to the references therein for a detailed presentation of these models and for their empirical justification. Several extensions, applications to a variety of financial problems, and rigorous justifications of the asymptotics can be found in [87, 88, 15, 16, 89].

According to the previous discussion, stochastic systems of the form (1), under some suitable assumptions implying ergodicity of the Y_t process, are appropriate for studying financial problems in this setting. Indeed, here the slow variables represent prices of assets or the wealth of the investor, whereas Y_t is an ergodic process representing the volatility and evolving on a faster time scale for δ small (and ε fixed).

On the other hand, Avellaneda et al. [7] used the theory of large deviations to give asymptotic estimates for the Black-Scholes implied volatility of option prices near maturity in models with constant volatility. In the recent paper [79], Feng, Fouque, and Kumar study the large deviations for system of the form (1) in the one-dimensional case $n = m = 1$, assuming that Y_t is an Ornstein-Uhlenbeck process and the coefficients in the equation for X_t do not depend on X_t. In their model ε represents a short maturity for the options, $1/\delta$ is the rate of mean reversion of Y_t, and the asymptotic analysis is performed for $\delta = \varepsilon^\alpha$ in the regimes $\alpha = 2$ and $\alpha = 4$. Their methods are based on the approach to large deviations developed in [80]. A related paper is [81] where the Heston model was studied in the regime $\delta = \varepsilon^2$ by methods different from [79]. Although sharing some motivations with [79] our results are quite different: we treat vector-valued processes with ϕ and σ depending on X_t in a rather general way and discuss all the three regimes (depending on the parameter α) in the period case and the regimes $\alpha \geq 2$ in the non compact case; our methods are also different, mostly from the theory of viscosity solutions for fully nonlinear PDEs and from the theory of homogenization and singular perturbations for such equations.

Large deviation principles have a large literature for diffusions with vanishing noise; some of them were extended to two-scale systems with small noise in the slow variables, see [131], [154], and more recently [122], [74], and [148]. Our methods can be also applied to this different scaling. The paper by Spiliopoulos [148] also states some results for the scaling of (1) under the assumptions of periodicity, but its methods based on weak convergence are completely different from ours. A related paper on homogenisation of a fully nonlinear PDE with vanishing viscosity is [52].

Part II-Neumann type problems for nonlocal equations with Lévy type terms

In the second part we present the results of [95] carried out during the period of research spent at the Laboratoire de Mathématique et Physique Théorique of Tours, under the supervision of Guy Barles.

We are interested in Neumann boundary value problems for partial integro-differential equations (PIDEs in short) related to discontinuous jump processes. In particular, the nonlocal terms are singular integral terms which arise when dealing with the infinitesimal generators of Lévy processes.

In the classical probabilistic approach to elliptic and parabolic partial differential equations, Neumann type boundary value problems are associated to stochastic processes being reflected on the boundary of the domain. The underlying idea is to force the stochastic process to remain inside the domain where the equation holds. This is obtained essentially by a reflection on the boundary, via the method developed by Lions and Sznitman [130].

In the classical (i.e. continuous) setting, a key result is, roughly speaking, the following: for a PDE with Neumann or oblique boundary conditions, there is a unique underlying reflection process and any consistent approximation will converge to it in the limit (see [130] and Barles, Lions [30]). At least in the case of normal reflections, this result is strongly connected to the study of the Skorohod problem and relies on the underlying stochastic processes being continuous.

In the case of discontinuous jump processes, the idea is the same but the situation is more complicated or, at least, the problem must be addressed in a different way. Indeed for jump processes which may exit the domain without having first hit the boundary, there are many ways to define a reflection. Also, because of the way the PIDE and the process are related, defining a reflection on the boundary will change the equation inside the domain. This is a new nonlocal phenomenon which is not encountered in the case of continuous processes and PDEs.

The general way to formulate the problem is to incorporate the reflection inside the definition of the nonlocal operator representing the infinitesimal generator, for example as follows

$$\mathscr{I}[u](x) = \int_{\mathbb{R}^n} u(x + j(x,z)) - u(x) d\mu(z), \tag{4}$$

where j is the so-called *function of jumps* satisfying a reflection condition preventing the process from leaving the domain, μ is a singular nonnegative Radon measure representing the intensity of the jumps from x to $x+z$ and satisfying some integrability condition and (4) has to be interpreted as a principal value $(P.V.)$ integral, that is

$$\mathscr{I}[u](x) = P.V. \int_{\mathbb{R}^n} u(x + j(x,z)) - u(x) d\mu(z) = \lim_{\delta \to 0^+} \int_{|z| \geq \delta} u(x + j(x,z)) - u(x) d\mu(z).$$

The jump function j satisfies a reflection condition, with respect to the domain Ω, of the following type

$$x + j(x,z) \in \bar{\Omega} \ \forall x \in \bar{\Omega}, \quad j(x,z) = z \text{ if } x + z \in \bar{\Omega}, \tag{5}$$

meaning that nothing happens and $j(x,z) = z$ if $x + z \in \bar{\Omega}$, while if $x + z \notin \bar{\Omega}$, then a "reflection" is performed in order to move the particle back to a point $x + j(x,z)$ inside.

Note that when $j(x,z)=z$, $\mathscr{I}[u]$ is the generator of a stochastic process which can jump from $x\in\bar{\Omega}$ to $x+z$ with a certain intensity, see e.g. [6],[62],[92]. Then, the choice of the jump function j influences the equation inside the domain.

In [24], Barles, Chasseigne, Georgeline, Jacobsen studied different models of reflection in the framework of Neumann boundary value problems for simple linear PIDEs in domains with flat boundary (namely the halfspace) of the following type

$$\begin{cases} u(x)-\mathscr{I}[u](x)+f(x)=0 & \text{in } \mathscr{H} \\ -\frac{\partial u}{\partial x_n}=0 & \text{on } \partial\mathscr{H}. \end{cases} \tag{6}$$

where $\mathscr{H}$ is the halfspace, i.e. $\mathscr{H}=\{x=(x_1,\cdots,x_n)\in\mathbb{R}^n, x_n>0\}$ and f is a bounded continuous function. We refer to Chapter 3 for a review of their main results.

Among the different models they consider, two types of reflections are particularly relevant for possible extensions to a more general setting. The first is the *normal projection*, close to the approach of Lions-Sznitman in [130], where outside jumps are immediately projected to the boundary by killing their normal component. This model has been thoroughly investigated in the paper [26] for fully non-linear equations set in general domains.

The second, the *censored* model, is the one we consider. In this case, any outside jump of the underlying process is cancelled (censored) and the process is restarted (resurrected) at the origin of that jump. The fact that the process is not allowed to jump outside $\bar{\Omega}$ is encoded in the definition of the nonlocal diffusion as follows

$$\mathscr{I}[u](x)=\lim_{\delta\to 0^+}\int_{\substack{|z|>\delta,\\ x+z\in\bar{\Omega}}}[u(x+z)-u(x)]d\mu(z), \tag{7}$$

where we remark that the domain of integration is restricted to the z such that $x+z\in\bar{\Omega}$, avoiding thus any outside jump.

We observe that throughout this part we interpret the equations in the sense of viscosity solution which provides a suitable definition of "generalized" Neumann boundary condition, in the sense that in certain cases the equation could hold up to the boundary and the Neumann condition would not be attained, and this corresponds to the fact that the underlying process could not reach the boundary. We refer to Chapter 4 for precise definitions of solutions.

In [24] it is shown, in the case of linear PIDEs as (6) that, the degree of singularity of μ influences the nature of the boundary value problem (6), in the sense that the Neumann condition is attained (in other words, the process hits the boundary) only if the measure is singular enough. When the singularity is of order stricly less than 1, e.g. when μ has density of the type

$$\frac{d\mu(z)}{dz}\sim\frac{1}{|z|^{n+\sigma}},\quad \sigma\in(0,1), \tag{8}$$

the process never reaches the boundary and equation holds up to the boundary.

On the other hand, when the singularity of the measure is strong, i.e when μ is of the type (4.3) with $\sigma \in [1,2)$, the situation is far more complicated, mainly due to the bad dependence in x of the operator in (7) and to the interplay between the singularity of the measure and the geometry of the boundary. We refer to Chapter 3 for some details on the case of more singular measures.

The aim of our work is the analysis of the well-posedness of censored type Neumann problems in the case of measures of singularity strictly less than 1 (namely with $\sigma \in (0,1)$ in (8)) in the presence of a Hamiltonian term, which forces the process to hit the boundary (and then Neumann condition be attained). Moreover, we consider general (enough smooth) domain for this kind of boundary value problems. To be more specific, we consider the following

$$\begin{cases} u(x) - \mathscr{I}[u](x) + H(x,Du) = 0 & \text{in } \Omega \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases} \tag{9}$$

where $H : \bar{\Omega} \times \mathbb{R}^n \mapsto \mathbb{R}$ is a continuous function, $\Omega \subset \mathbb{R}^n$ is an open (smooth enough) domain and $\mathscr{I}[u]$ is an *integro-differential operator of censored type and of order stricly less than* 1 defined as

$$\mathscr{I}[u](x) = P.V. \int_{x+j(x,z)\in\bar{\Omega}} [u(x+j(x,z)) - u(x)]d\mu_x(z) \tag{10}$$

where μ_x is a nonnegative Radon measure with density of the type

$$\frac{d\mu_x}{dz} = g(x,z)|z|^{-(n+\sigma)} \quad \sigma \in (0,1), \tag{11}$$

for a nonnegative bounded function g, Lipschitz in x uniformly with respect to z and $j(x,z)$ are more general *jump functions* $j(x,z)$ than in [24] (see Chapter 4 Section 4.2, assumptions (M0),(M1), (J0), (J1), (J2)). We remark that (10) is said to be of *order strictly less than* 1 with reference to the condition $\sigma \in (0,1)$ in (11).

Note that, since censored type processes are not allowed to jump outside $\bar{\Omega}$, we don't need any condition on Ω^c in the boundary value problem (9).

We remark that we follow the PIDE analytical approach developed in [24], in the sense that we work directly with the infinitesimal generator and not with the reflected process. We refer to the introduction of [24] for more details and probabilistic references on censored processes.

We consider a class of Hamiltonians with a gradient growth stronger than the diffusive term in the nonlocal operator. The first example is a Hamiltonian H with *superfractional coercive* growth in the gradient variable, namely

$$H(x,p) = a(x)|p|^m - f(x), \tag{12}$$

where $m > \sigma$, $a, f : \bar{\Omega} \mapsto \mathbb{R}$ are bounded and continuous functions and $a(x) \geq a_0 > 0$ for some fixed constant a_0. We remark that the positivity of a and the condition $m > \sigma$ make the first-order term the leading term in the equation. We also observe that we have no other additional restriction on m (in particular, we can deal with Hamiltonians as in (12) with $m < 1$), allowing the study of Hamiltonians which are concave in Du.

The second main example is a Hamiltonian H of *Bellman type*, which arises in the study of Hamilton-Jacobi equations associated to optimal exit time problems, such as

$$H(x,p) = \sup_{\alpha \in \mathscr{A}} \{-b(x,\alpha) \cdot p - l(x,\alpha)\}, \tag{13}$$

where $\mathscr{A}$ is a compact metric space (the control space) and b, l are continuous and bounded functions (we refer the reader to [14] and [85] for some connections between this type of equations and control problems). Note that the diffusive term of $\mathscr{I}$ defined in (10) is of lower order than the first-order term when we assume $\sigma < 1$. We also observe that, as in [34] and [152], the well-posedness of (9) with Hamiltonian as in (13) is based on a careful study of the effects of the drift b at each point of $\partial\Omega$.

Our main result is a comparison principle between bounded viscosity sub and supersolutions to (9), namely Theorem 4.2.6 proved in Chapter 4, Section 4.2. We remark that the proof of this result is not standard even in the case $\sigma < 1$ in domains with flat boundary (namely, the halfspace). The difficulties are mainly due to the fact that operators as in (10) behave badly in x. The main idea which is behind the proof is to localize the argument on points which have the same distance from the boundary and this is carried out through the use of a non-standard non regular test function. After the localization procedure, the rest of the proof in the case of the halfspace is simple, whereas in the case of general domains, a lot of technical difficulties arise form the way the x-depending set of integration of $\mathscr{I}$ interferes with the geometry of the boundary. To face these extra technical difficulties, we rectify the boundary relying on the smoothness of Ω.

As the first main application of our results, we get existence and uniqueness for (9) by standard Perron's method. We refer to Chapter 4, Section 4.2.

Finally, in Chapter 5, we present some applications of our results to the evolutive setting, such as existence, uniqueness, and the asymptotic behaviour as $t \to +\infty$ of the solution of the associated Cauchy problem.

Part III-Quantitative Borell-Brascamp-Lieb inequalities (for power concave functions), and applications to isoperimetric inequalities for some variational functionals

This part contains the results of [97] carried out in collaboration with Paolo Salani (University of Florence).

Our main results are some refinements of a class of integral inequalities known as *Borell-Brascamp-Lieb* inequalities (BBL in the following).

Before recalling the BBL inequality, we first introduce some notations. We denote by u_0 and u_1 real non-negative bounded functions belonging to $L^1(\mathbb{R}^n)$ $(n \geq 1)$ with compact support Ω_0 and Ω_1 respectively. Moreover, let

$$I_i = \int_{\Omega_i} u_i dx.$$

For $\lambda \in (0,1)$, let Ω_λ be the Minkowski convex combination (with coefficient λ) of Ω_0 and Ω_1, that is

$$\Omega_\lambda = (1-\lambda)\Omega_0 + \lambda\Omega_1 = \{(1-\lambda)x_0 + \lambda x_1 \, : \, x_0 \in \Omega_0, x_1 \in \Omega_1\}$$

where $+$ denotes the vector sum. For $q \in [-\infty, +\infty]$ and $\mu \in (0,1)$ we denote by $\mathscr{M}_q(a,b,\mu)$ the (μ-weighted) *q-mean* of two non-negative numbers a and b, which is defined as follows:

$$\mathscr{M}_q(a,b;\mu) = \begin{cases} \max\{a,b\} & q = +\infty \\ [(1-\mu)a^q + \mu b^q]^{\frac{1}{q}} & \text{if } 0 \neq q \in \mathbb{R} \text{ and } ab > 0 \\ a^{1-\mu} b^\mu & \text{if } q = 0 \\ \min\{a,b\} & q = -\infty \\ 0 & \text{when } q \in \mathbb{R} \text{ and } ab = 0\,. \end{cases} \tag{14}$$

Note that the arithmetic mean and geometric mean corresponds to the $q = 1$ and $q = 0$, respectively. We recall the BBL inequality:

Theorem (BBL inequality). *Let* $0 < \lambda < 1, -\frac{1}{n} \leq p \leq \infty$, $0 \leq h \in L^1(\mathbb{R}^n)$ *and assume*

$$h((1-\lambda)x + \lambda y) \geq \mathscr{M}_p(u_0(x), u_1(y), \lambda),$$

for every $x \in \Omega_0, y \in \Omega_1$. *Then*

$$\int_{\Omega_\lambda} h(x)\, dx \geq \mathscr{M}_{\frac{p}{np+1}}(I_0, I_1, \lambda)\,. \tag{15}$$

Here the number $p/(np+1)$ has to be interpreted in the obvious way in the extremal case, i.e. it is equal to $-\infty$ when $p = -1/n$ and to $1/n$ when $p = \infty$.

The BBL inequality was first proved in a slightly different form for $p > 0$ by Henstock and Macbeath (with $n = 1$) in [106] and by Dinghas in [72]. In its generality it is stated and proved by Brascamp and Lieb in [47] and by Borell in [42] and the equality conditions are discussed in [73]. Since the equality case is rather complicated to state, we refer for the precise statement to [73]. Roughly speaking, equality holds in (15) if and only if the functions $u_i, i = 0, 1, 2$ are almost everywhere equal to some suitable homotheties of the same convex function.

The case $p = 0$ was previously proved by Prékopa [138] and Leindler [124] (and rediscovered by Brascamp and Lieb in [46]) and it is usually known as the *Prékopa-Leindler inequality* (PL inequality in the following). It is worth to remark that the PL inequality (and then the BBL inequality, for every p) can be considered as a functional form of the *Brunn-Minkowski inequality*, which in its classical form states that if Ω_1, Ω_0 are two nonempty compact convex sets of $\mathbb{R}^n$ and $\lambda \in (0, 1)$, then

$$|(1-\lambda)\Omega_0 + \lambda\Omega_1| \geq \mathscr{M}_{1/n}(|\Omega_0|, |\Omega_1|, \lambda)$$

and equality holds precisely when Ω_0 and Ω_1 are equal up to translation and dilatation. A generalization to measurable subsets of $\mathbb{R}^n$ has been proved later in [133] and [102]. For more details on the Brunn-Minkowski inequality we refer to Section 6.2 and to [93] as a general and exhaustive reference.

In this part of the thesis we are interested in the investigation of stability problems for the BBL inequality. The typical kind of questions we aim at answering is the following: if $\int_{\Omega_\lambda} h(x)\, dx$ “approximately ”coincides with $\mathscr{M}_{\frac{p}{np+1}}(I_0, I_1, \lambda)$, can we infer some kind of “closeness ” of the functions u_0, u_1 to the equality condition?

The first step when dealing with a stability problem is to give a precise meaning to the previous question, i.e. choose a measure of the “closeness” of the functions to the equality condition. Depending on the choice of the measure, different kind of results can be obtained.

Significant interest has recently arisen towards the stability of the Brunn-Minkowski inequality and several kind of results have been obtained depending on different measures, see [99, 83, 56, 57, 75, 82]. Concerning the PL inequality, the investigation of stability questions has been recently started by Ball and Böröczky in [12, 13]. Note that all these results are in [48] and all these results are written in terms of the L^1 distance between the involved functions.

Our main achievements are stability results for the BBL inequality in terms of some distance between the support sets Ω_0 and Ω_1 of u_0 and u_1 and some consequent “quantitative” versions of the BBL inequality. With the adjective “quantitative”, we mean that we strengthen (15) in terms of some distance between the support sets Ω_0 and Ω_1 of u_0 and u_1.

The quantitative versions we give are mainly of two types. The first is written in terms of the Hausdorff distance between (two suitable homothetic copies of) Ω_0 and Ω_1. We recall that the Hausdorff distance $H(K, L)$ between two sets $K, L \subseteq \mathbb{R}^n$ is defined as follows:

$$H(K,L) := \inf\{r \geq 0 : K \subseteq L + r\overline{B}_n, K \subseteq L + r\overline{B}_n\},$$

where $B_n = \{x \in \mathbb{R}^n : |x| < 1\}$ is the (open) unit ball in $\mathbb{R}^n$. Then we set

$$H_0(K,L) = H(\tau_0 K, \tau_1 L),$$

where τ_1, τ_0 are two homotheties (i.e. translation plus dilation) such that $|\tau_0 K| = |\tau_1 L| = 1$ and such that the centroids of $\tau_0 K$ and $\tau_1 L$ coincide (by centroid we mean the geometric center, i.e. the average of all the points of the set).

Under the assumptions of Theorem (BBL inequality) (and assuming some concavity property of the functions u_0, u_1, namely p-concavity, for the definition see (6.5), Chapter 6, Section 6.1) we prove that

$$\int_{\Omega_\lambda} h(x)\,dx \geq \mathscr{M}_{\frac{p}{np+1}}(I_0, I_1, \lambda) + \beta\, H_0(\Omega_0, \Omega_1)^{\frac{(n+1)(p+1)}{p}}, \tag{16}$$

where β is a constant depending only on n, λ, p, I_0, I_1, the diameters and the measures of Ω_0 and Ω_1.

We provide another quantitative result analogous to (16), but in this case the quantitative terms depends on the *relative asymmetry* (or *Fraenkel asymmetry*) of Ω_0 and Ω_1; we recall that the relative asymmetry of two sets K and L is defined as follows

$$A(K,L) := \inf_{x \in \mathbb{R}^n} \left\{ \frac{|K \Delta (x + \lambda L)|}{|K|}, \lambda = \left(\frac{|K|}{|L|}\right)^{\frac{1}{n}} \right\},$$

where, for $\Omega \subseteq \mathbb{R}^n$, $|\Omega|$ denotes its Lebesgue measure, while Δ denotes the operation of symmetric difference, i.e. $\Omega \Delta B = (\Omega \setminus B) \cup (B \setminus \Omega)$.

For the precise statement of the above quantitative results and for some other relevant remarks on the assumptions and proofs, and for explicit values of the constants, we refer to Chapter 6, Section 6.1.

Note that our results (as (16)) are written as quantitative forms of the involved inequalities, but they can be obviously interpreted also as stability results for the same inequalities.

The crucial part in the proofs of the above stated results relies on an estimate of the measures of the supports sets of the involved functions; this estimate is contained in Theorem 7.0.1 (Chapter 7), which can be in fact considered our main result. There we prove that if we are close to equality in (15), then the measure of $(1-\lambda)\Omega_0 + \lambda\Omega_1$ is close to $\mathscr{M}_{1/n}(|\Omega_0|, |\Omega_1|, \lambda)$. Therefore, once Theorem 7.0.1 is proved, the proofs follow by applying different quantitative versions of the classical Brunn-Minkowski inequality (namely [99] and [83], see Chapter 6, Section 6.2). We note also that further recent stability/quantitative results for the Brunn-Minkowski inequality are contained in [56, 57, 75, 82]. A combination of these with Theorem 7.0.1 would lead to further stability/quantitative theorems for the BBL inequality, which could be an interesting topic for further research.

As applications of the above results, we can derive interesting quantitative versions of some interpolation inequalities for functionals that can be written in terms of the solutions of suitable elliptic boundary value problem. This is in fact the original reason for which we tackled the stability of the BBL inequality. We present these results in Chapter 8.

For the sake of simplicity and clearness of exposition, as a toy model we will analyse in details the torsion problem, that is

$$\begin{cases} \Delta u = -2 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{17}$$

We recall that the torsional rigidity $\tau(\Omega)$ of Ω is defined as follows

$$\frac{1}{\tau(\Omega)} = \inf\left\{ \frac{\int_\Omega |Dw|^2\,dx}{(\int_\Omega |w|\,dx)^2} \;:\; w \in W_0^{1,2}(\Omega),\ \int_\Omega |w|\,dx > 0 \right\}$$

and that, in general, when a solution u to problem (17) exists, we have

$$\tau(\Omega) = \frac{(\int_\Omega |u|\,dx)^2}{\int_\Omega |Du|^2\,dx} = \int_\Omega u\,dx\,.$$

It is well-known that τ satisfies the following Brunn-Minkowski type inequality on the class of compact convex sets with non-empty interior:

$$\tau((1-\lambda)\Omega_0 + \lambda\,\Omega_1) \geq \mathscr{M}_{\frac{1}{n+2}}\left(\tau(\Omega_0), \tau(\Omega_1), \lambda\right), \tag{18}$$

where equality holds in (18) if and only if Ω_0 and Ω_1 coincide up to a homothety. Inequality (18) was proved by Borell in [44] and the equality conditions are provided in [59]. However (18) can also be seen as a consequence of the BBL inequality and therefore we can apply the quantitative results for the BBL inequality to get corresponding refinements for (18). We present these results in Theorem 8.1.4, Chapter 8.

Furthermore, the existing literature (see [40, Remark 6.1] and [49, Proposition 4.1]) shows that it is possible to use (18) to prove the following Urysohn's type inequality for the torsional rigidity

$$\tau(\Omega) \leq \tau(\Omega^\sharp) \qquad \text{for every convex set } \Omega\,, \tag{19}$$

where $\Omega^\sharp$ is the ball with the same mean-width of Ω. In other words, the previous inequality can be rephrased as follows: *among convex sets with given mean width, the torsional rigidity is maximized by balls*.

The main results of this part of the thesis are two refinements of (19) on the flavour of the quantitative results stated above for the BBL, one in terms of the Hausdorff distance of Ω from $\Omega^\sharp$ and another one in terms of the relative asymmetry of Ω. For the precise statements see Theorem 8.1.6 stated in Chapter 8.

We remark that similar results can be obtained in a suitable form for many other variational functionals with similar properties as τ and satisfying suitable Brunn-Minkowski inequalities, where nonlinear operators are involved as the q-Laplacian, the Finsler laplacian and nonlinear operators not in divergence form as the Pucci Extremal operators. We will see in Chapter 8, Section 8.2 some general results regarding the stability of the so called *mean width rearrangements*, a new kind of rearrangement recently introduced by Salani in [145], which in fact include most of the examples we can manage with this method.

Part I

Large deviations of some stochastic volatility models by viscosity methods

Chapter 1

Periodic case

1.1 Introduction

In this chapter we study the asymptotic behaviour as $\varepsilon \to 0$ of stochastic systems of the following kind

$$\begin{cases} dX_t = \varepsilon\phi(X_t,Y_t)dt + \sqrt{2\varepsilon}\sigma(X_t,Y_t)dW_t & X_0 = x \in \mathbb{R}^n, \\ dY_t = \varepsilon^{1-\alpha}b(Y_t)dt + \sqrt{2\varepsilon^{1-\alpha}}\tau(Y_t)dW_t & Y_0 = y \in \mathbb{R}^m, \end{cases} \tag{1.1}$$

where $\varepsilon > 0$, W_t is a standard r-dimensional Brownian motion, the matrix τ is non-degenerate and the coefficients of the system are periodic with respect to the variable y.

We consider a functional of the trajectories of (1.1) of the form

$$v^\varepsilon(t,x,y) := \varepsilon \log E\left[e^{h(X_t)/\varepsilon} | (X.,Y.) \text{ satisfy (1.1)}\right], \tag{1.2}$$

where h is a bounded continuous function. We observe that the logarithmic form of this payoff is motivated by the applications to large deviations that we want to give.

We note that v^ε can be characterized as the solution of the Cauchy problem with initial data $v^\varepsilon(0,x,y) = h(x)$ for a fully nonlinear parabolic equation in $n+m$ variables (see Proposition 1.1.2 where we recall this result).

Our first aim is to prove that, under suitable assumptions, the functions $v^\varepsilon(t,x,y)$ converge to a function $v(t,x)$ characterized as the solution of the Cauchy problem for a first order Hamilton-Jacobi equation in n space dimensions

$$v_t - \bar{H}(x,Dv) = 0 \text{ in }]0,T[\times\mathbb{R}^n, \quad v(0,x) = h(x), \tag{1.3}$$

for a suitable effective Hamiltonian $\bar{H}$.

We observe that the existing techniques to treat this kind of problems have been developped so far mainly under assumptions implying some kind of boundedness of the fast variable. We refer mainly to the methods of [4], stemming from the pioneering paper of Lions-Papanicolau-Varadhan [128] and Evans' [76] on periodic homogenization and its extensions to singular perturbations [1–3].

A classical hypothesis is the periodicity with respect to Y_t of the coefficients of the stochastic system, which in particular implies the periodicity in y of the solutions v^{ε}. Throughtout this chapter we carry on our analysis under this assumption, treating what we call the *periodic case*.

In the periodic case, the convergence is quite standard once the effective problem is identified and a comparison principle is proved. The most significant part is the identification of the *effective Hamiltonian* $\bar{H}$, which is obtained by solving a suitable cell problem.

We indentify three different cases, depending on α, the supercritical case ($\alpha > 2$), the critical case ($\alpha = 2$) and the subcritical case ($\alpha < 2$). In all the three cases, we solve the cell problem, we identify the limit Hamiltonian $\bar{H}$ and we represent it through explicit formulas. In particular, in the supercritical case the cell problem involves a linear elliptic operator and $\bar{H}$ can be written in terms of the invariant measure of the process

$$dY_t = b(Y_t)dt + \sqrt{2}\tau(Y_t)dW_t.$$

In the critical case the cell problem is a fully nonlinear elliptic PDE and $\bar{H}$ can be represented in various ways based, e.g., on stochastic control. Finally, in the subcritical case the cell problem is of first order and nonlinear, and a representation formula for $\bar{H}$ can be given in terms of deterministic control.

The main application of the convergence results is a large deviations analysis of (1.1). In particular, we prove for the three different regimes that the measures associated to the process X_t in (1.1) satisfy a Large Deviation Principle and we find representation formulas for the rate function. Following [79] we also apply this result to derive an estimate of option prices near maturity and an asymptotic formula for the implied volatility.

We recall that in the recent paper [79], Feng, Fouque, and Kumar study the large deviations for system of the form (1.1) in the one-dimensional case $n = m = 1$, assuming that Y_t is an Ornstein-Uhlenbeck process and the coefficients in the equation for X_t do not depend on X_t and they perform the asymptotic analysis in the regimes $\alpha = 2$ and $\alpha = 4$. We also remark that their methods are based on the approach to large deviations developed in [80]. A related paper is [81] where the Heston model was studied in the regime $\delta = \varepsilon^2$ by methods different from [79]. Although sharing some motivations with [79] our results are quite different: we treat vector-valued processes with ϕ and σ depending on X_t in a rather general way and discuss all the three regimes depending on the parameter α; our methods are also different, mostly from the theory of viscosity solutions for fully nonlinear PDEs and from the theory of homogenization and singular perturbations for such equations.

Large deviation principles have a large literature for diffusions with vanishing noise; some of them were extended to two-scale systems with small noise in the slow variables, see

[131], [154], and more recently [122], [74], and [148]. We remark that our methods can be also applied to this different scaling. We recall the paper by Spiliopoulos [148], where some results for the scaling of (1.1) are stated under the assumptions of periodicity, but its methods based on weak convergence are completely different from ours. A related paper on homogenization of a fully nonlinear PDE with vanishing viscosity is [52].

1.1.1 The stochastic volatility model

We consider fast stochastic volatility systems that can be written in the form

$$\begin{cases} dX_t = \phi(X_t,Y_t)dt + \sqrt{2}\sigma(X_t,Y_t)dW_t, & X_0 = x \in \mathbb{R}^n \\ dY_t = \varepsilon^{-\alpha} b(Y_t)dt + \sqrt{2\varepsilon^{-\alpha}}\tau(Y_t)dW_t, & Y_0 = y \in \mathbb{R}^m. \end{cases} \tag{1.4}$$

where $\varepsilon > 0$, $\alpha > 1$ and W_t is a r-dimensional standard Brownian motion. We recall that we denote by $\mathbf{M}^{n,r}$ the set of $n \times r$ matrices.

We assume $\phi : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n, \sigma : \mathbb{R}^n \times \mathbb{R}^m \to \mathbf{M}^{n,r}$are bounded continuous functions, Lipschitz continuous in (x,y) and periodic in y. Moreover $b : \mathbb{R}^m \to \mathbb{R}^m, \tau : \mathbb{R}^m \to \mathbf{M}^{m,r}$ are locally Lipschitz continuous functions, periodic in y. Note that in this part we will denote by $\mathbb{T}^m$ the m-dimensional torus. These assumptions and notations will hold throughout the chapter. We assume the uniform nondegeneracy of the diffusion driving the fast variable Y_t, i.e., for some $\theta > 0$

$$\xi^T \tau(y)\tau(y)^T \xi = |\tau^T(y)\xi|^2 > \theta|\xi|^2 \quad \text{for every } y \in \mathbb{R}, \xi \in \mathbb{R}^m. \tag{1.5}$$

In order to study small time behavior of the system (1.4), we rescale time $t \to \varepsilon t$ for $0 < \varepsilon \ll 1$, so that the typical maturity will be of order of ε. Denoting the rescaled processes by X_t^ε and Y_t^ε we get

$$\begin{cases} dX_t^\varepsilon = \varepsilon\phi(X_t^\varepsilon,Y_t^\varepsilon)dt + \sqrt{2\varepsilon}\sigma(X_t^\varepsilon,Y_t^\varepsilon)dW_t, & X_0^\varepsilon = x \in \mathbb{R}^n \\ dY_t^\varepsilon = \varepsilon^{1-\alpha} b(Y_t^\varepsilon)dt + \sqrt{2\varepsilon^{1-\alpha}}\tau(Y_t^\varepsilon)dW_t, & Y_0^\varepsilon = y \in \mathbb{R}^m. \end{cases} \tag{1.6}$$

1.1.2 The logarithmic transformation and the HJB equation

We consider the functional

$$u^\varepsilon(t,x,y) := E\left[g(X_t^\varepsilon) \,|\, (X_s^\varepsilon, Y_s^\varepsilon) \text{ satisfy (1.6) for } s \in [0,t]\right] \tag{1.7}$$

where $g \in BC(\mathbb{R}^n)$. We recall that $BC(\mathbb{R}^n)$ is the space of bounded continuous functions in $\mathbb{R}^n$.

The partial differential equation associated to the functions u^ε is

$$u_t - \varepsilon \mathrm{tr}(\sigma\sigma^T D^2_{xx}u) - \varepsilon\phi \cdot D_x u - 2\varepsilon^{1-\frac{\alpha}{2}} \mathrm{tr}(\sigma\tau^T D^2_{xy}u)$$
$$- \varepsilon^{1-\alpha} b \cdot D_y u - \varepsilon^{1-\alpha} \mathrm{tr}(\tau\tau^T D^2_{yy}u) = 0 \quad (1.8)$$

in $(0,T) \times \mathbb{R}^n \times \mathbb{R}^m$, where b and τ are computed in y, ϕ and σ are computed in (x,y). The equation is complemented with the initial condition:

$$u(0,x,y) = g(x).$$

Remark 1.1.1. Note that, since we assume the periodicity in y of the coefficients of the equation b, σ, τ, ϕ, we have that the solution u^ε of the equation (1.8) is periodic in y itself.

We introduce the logarithmic transformation method (see [85]). Assume that

$$g(x) = e^{h(x)/\varepsilon} \text{ with } h \in BC(\mathbb{R}^n)$$

and define

$$v^\varepsilon(t,x,y) := \varepsilon \log u^\varepsilon = \varepsilon \log E\left[e^{h(X^\varepsilon_t)/\varepsilon} \,|\, (X^\varepsilon_s, Y^\varepsilon_s) \text{ satisfy (1.6) for } s \in [0,t]\right], \quad (1.9)$$

where u^ε is defined in (1.7), $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, and $t \geq 0$. By (1.8) and some computations, one sees that the equation associated to v^ε is

$$v_t = |\sigma^T D_x v|^2 + \varepsilon \mathrm{tr}(\sigma\sigma^T D^2_{xx}v) + \varepsilon\phi \cdot D_x v + 2\varepsilon^{-\frac{\alpha}{2}} (\tau\sigma^T D_x v) \cdot D_y v +$$
$$2\varepsilon^{1-\frac{\alpha}{2}} \mathrm{tr}(\sigma\tau^T D^2_{xy}v) + \varepsilon^{1-\alpha} b \cdot D_y v + \varepsilon^{-\alpha} |\tau^T D_y v|^2 + \varepsilon^{1-\alpha} \mathrm{tr}(\tau\tau^T D^2_{yy}v), \quad (1.10)$$

where b and τ are computed in y, ϕ and σ are computed in (x,y). In general, the functions u^ε are not smooth but one can check that v^ε is a viscosity solutions of (1.10) (see in particular Chapter VI and VII of [85]).

In the following proposition we characterize the value function v^ε as the unique continuous viscosity solution to a suitable parabolic problem with initial data for each of the three regimes. A general reference for these issue is [85]. The equation (1.10) satisfied by v^ε involves a quadratic nonlinearity in the gradient. This case was studied by Da Lio and Ley in [69], where the reader can find a proof of the next result.

Proposition 1.1.2. *i) Let $\alpha \geq 2$ and define*

$$H^\varepsilon(x,y,p,q,X,Y,Z) \;:=\; |\sigma^T p|^2 + b \cdot q + tr(\tau\tau^T Y) + \varepsilon\left(tr(\sigma\sigma^T X) + \phi \cdot p\right)$$
$$+\; 2\varepsilon^{\frac{\alpha}{2}-1} (\tau\sigma^T p) \cdot q + 2\varepsilon^{\frac{1}{2}} tr(\sigma\tau^T Z) + \varepsilon^{\alpha-2} |\tau^T q|^2.$$

Then v^ε is the unique bounded continuous viscosity solution of the Cauchy problem

$$\begin{cases} \partial_t v^\varepsilon - H^\varepsilon\left(x,y,D_x v^\varepsilon, \frac{D_y v^\varepsilon}{\varepsilon^{\alpha-1}}, D^2_{xx}v^\varepsilon, \frac{D^2_{yy}v^\varepsilon}{\varepsilon^{\alpha-1}}, \frac{D^2_{xy}v^\varepsilon}{\varepsilon^{\frac{\alpha-1}{2}}}\right) = 0 & in\ [0,T]\times\mathbb{R}^n\times\mathbb{R}^m, \\ v^\varepsilon(0,x,y) = h(x) & in\ \mathbb{R}^n\times\mathbb{R}^m. \end{cases} \tag{1.11}$$

ii) Let $\alpha < 2$ and define

$$\begin{aligned} H_\varepsilon(x,y,p,q,X,Y,Z) \ &:= \ |\sigma^T p|^2 + |\tau^T q|^2 + 2(\tau\sigma^T p)\cdot q + \varepsilon\left(tr(\sigma\sigma^T X) + \phi\cdot p\right) \\ &+ \ \varepsilon^{1-\frac{\alpha}{2}}(b\cdot q + tr(\tau\tau^T Y)) + 2\varepsilon^{1-\frac{\alpha}{4}} tr(\sigma\tau^T Z). \end{aligned}$$

Then v^ε is the unique bounded continuous viscosity solution of the Cauchy problem

$$\begin{cases} \partial_t v^\varepsilon - H_\varepsilon\left(x,y,D_x v^\varepsilon, \frac{D_y v^\varepsilon}{\varepsilon^{\frac{\alpha}{2}}}, D^2_{xx}v^\varepsilon, \frac{D^2_{yy}v^\varepsilon}{\varepsilon^{\frac{\alpha}{2}}}, \frac{D^2_{xy}v^\varepsilon}{\varepsilon^{\frac{\alpha}{4}}}\right) = 0 & in\ [0,T]\times\mathbb{R}^n\times\mathbb{R}^m, \\ v^\varepsilon(0,x,y) = h(x) & in\ \mathbb{R}^n\times\mathbb{R}^m. \end{cases} \tag{1.12}$$

Our goal is to study the limit as $\varepsilon \to 0$ of the functions v^ε described in Proposition 1.1.2. Following the viscosity solution approach to singular perturbation problems (see [3],[2]), we define a limit or effective Hamiltonian $\bar{H}$ and we characterize the limit of v^ε as the unique solution of an appropriate Cauchy problem with Hamiltonian $\bar{H}$. The first step in the procedure is the identification of the limit Hamiltonian. In order to define this operator, we make the ansatz that the function v^ε admits the formal asymptotic expansion

$$v^\varepsilon(t,x,y) = v^0(t,x) + \varepsilon^{\alpha-1}w(t,x,y) \tag{1.13}$$

and plug it into the equation. In the following sections we show that the limit Hamiltonian is different in the three different regimes: the critical case ($\alpha = 2$), the supercritical case (when $\alpha > 2$), and the subcritical case (when $\alpha < 2$).

Remark 1.1.3. Numerical experiments in [150] indicate that the first order approximation in the expansion (1.13) is sufficiently accurate to find option prices in a fast mean-reversion case of the volatility process.

1.2 The critical case: $\alpha = 2$

1.2.1 The ergodic problem and the effective Hamiltonian

Equation (1.10) with $\alpha = 2$ becomes

$$v_t = |\sigma^T D_x v|^2 + \varepsilon \mathrm{tr}(\sigma\sigma^T D^2_{xx} v) + \varepsilon\phi \cdot D_x v + \frac{2}{\varepsilon}(\tau\sigma^T D_x v)\cdot D_y v \qquad (1.14)$$
$$+2\mathrm{tr}(\sigma\tau^T D^2_{xy} v) + \frac{1}{\varepsilon} b \cdot D_y v + \frac{1}{\varepsilon^2}|\tau^T D_y v|^2 + \frac{1}{\varepsilon}\mathrm{tr}(\tau\tau^T D^2_{yy} v).$$

We plug in the equation (1.14) the formal asymptotic expansion

$$v^\varepsilon(t,x,y) = v^0(t,x) + \varepsilon w(t,x,y)$$

and we obtain

$$v^0_t - |\sigma^T D_x v^0|^2 - 2(\tau\sigma^T D_x v^0)\cdot D_y w - b\cdot D_y w - |\tau^T D_y w|^2 - \mathrm{tr}(\tau\tau^T D^2_{yy} w) = O(\varepsilon).$$

We want to eliminate the function w, usually called the *corrector*, and the dependence on y in this equation and remain with a left hand side of the form $v^0_t - \bar{H}(x, D_x v^0)$. Therefore we freeze $\bar{x}$ and $\bar{p} = D_x v^0(\bar{x})$ and define the *effective Hamiltonian* $\bar{H}(\bar{x},\bar{p})$ as the unique constant such that the following stationary PDE in $\mathbb{R}^m$, called *cell problem*, has a viscosity solution w:

$$\bar{H}(\bar{x},\bar{p}) - |\sigma^T \bar{p}|^2 - 2(\tau\sigma^T\bar{p})\cdot D_y w(y) - b\cdot D_y w(y) - |\tau^T D_y w(y)|^2 - \mathrm{tr}(\tau\tau^T D^2_{yy} w(y)) = 0, \qquad (1.15)$$

where σ is computed in $(\bar{x}, y)$ and τ, b in y. This is an additive eigenvalue problem that arises the theory of ergodic control and has a wide literature. Under our standing assumptions we have the following result.

Proposition 1.2.1. *For any fixed $(\bar{x},\bar{p})$, there exists a unique $\bar{H}(\bar{x},\bar{p})$ for which the equation* (1.15) *has a periodic viscosity solution w. Moreover $w \in C^{2,\alpha}$ for some* $0 < \alpha < 1$ *and satisfies for some $C > 0$ independent of $\bar{p}$ and $\forall \bar{x},\bar{p} \in \mathbb{R}^n$*

$$\max_{y\in\mathbb{R}^m} |Dw(y;\bar{x},\bar{p})| \le C(1+|\bar{p}|). \qquad (1.16)$$

To prove Proposition 1.2.1, we need the following lemma. We will use the small discount approximation for $\delta > 0$

$$\delta w_\delta + F(\bar{x}, y, \bar{p}, Dw_\delta, D^2 w_\delta) - |\sigma(\bar{x},y)\bar{p}|^2 = 0, \qquad (1.17)$$

where

$$F(\bar{x}, y, \bar{p}, q, Y) := -\mathrm{tr}(\tau\tau^T(y)Y) - |\tau^T(y)q|^2 - b(y)\cdot q - 2(\tau(y)\sigma^T(\bar{x}, y)\bar{p})\cdot q. \tag{1.18}$$

Lemma 1.2.2. *Let $\delta > 0$ and $w_\delta(\cdot;\bar{x},\bar{p}) \in C^2(\mathbb{R}^m)$ be a periodic solution of* (1.17)*. Then there exists $C > 0$ independent of $\bar{p}$ such that for all $\bar{x}, \bar{p} \in \mathbb{R}^n$ it holds*

$$\max_{y\in\mathbb{R}^m} |D_y w_\delta(y;\bar{x},\bar{p})| \leq C(1+|\bar{p}|). \tag{1.19}$$

Proof of Lemma 1.2.2. We remark that analogous results for similar equations have been proved by Lions and Souganidis in [129]. For the sake of completeness we give the following proof, which uses the Bernstein method, following the derivation of similar estimates in [78]. We carry out the computations in the case τ, σ, b are C^1. When τ, σ, b are Lipschitz the result can be proved by smooth approximation.

Throughout the proof for simplicity of notations we denote by w_i and w_{ji} respectively the derivative of w with respect to the i-th variable and the derivative of w_i with respect to the j-th variable.

Denote by $w^\delta := w_\delta(y;\bar{x},\bar{p})$ the solution of (1.17). By comparison with constant sub- and supersolutions we get the uniform bound

$$|\delta w^\delta| \leq \max_{y\in\mathbb{R}^m} |\sigma^T(\bar{x},y)\bar{p}|^2 \quad \forall y \in \mathbb{R}^m. \tag{1.20}$$

Define the function z as follows

$$z := |Dw^\delta|^2.$$

Should z attains its maximum at some point y_0, then at y_0

$$z_i = 2w_k^\delta w_{ki}^\delta = 0 \quad i = 1,\dots,m, \tag{1.21}$$

where we are adopting the summation convention, and

$$0 \leq -(\tau\tau^T)_{ij} z_{ij} = -2(\tau\tau^T)_{ij} w_{ki}^\delta w_{kj}^\delta - 2w_k^\delta(\tau\tau^T)_{ij} w_{ijk}^\delta. \tag{1.22}$$

Then at y_0

$$\theta|D^2 w^\delta|^2 \leq (\tau\tau^T)_{ij} w_{ki}^\delta w_{kj}^\delta \leq -w_k^\delta(\tau\tau^T)_{ij} w_{ijk}^\delta = -w_k^\delta\left((\tau\tau^T)_{ij} w_{ij}^\delta\right)_k + w_k^\delta(\tau\tau^T)_{ij,k} w_{ij}^\delta,$$

where we have used (1.22). Thus at y_0

$$\theta|D^2 w^\delta|^2 \leq w_k^\delta\left(-\delta w^\delta + (2\tau\sigma^T\bar{p}+b)\cdot Dw^\delta + |\tau^T Dw^\delta|^2 + |\sigma^T\bar{p}|^2\right)_k + w_k^\delta(\tau\tau^T)_{ij,k} w_{ij}^\delta,$$

where we have used (1.17). Thanks to (1.21)

$$w_k^\delta(|\tau^T Dw^\delta|^2)_k = w_k^\delta((\tau\tau^T)_{ij}w_i^\delta w_j^\delta)_k = \\ w_k^\delta(\tau\tau^T)_{ij,k}w_i^\delta w_j^\delta + w_k^\delta(\tau\tau^T)_{ij}w_{ik}^\delta w_j^\delta + w_k^\delta(\tau\tau^T)_{ij}w_i^\delta w_{jk}^\delta = w_k^\delta(\tau\tau^T)_{ij,k}w_i^\delta w_j^\delta.$$

Moreover

$$w_k^\delta(\tau\tau^T)_{ij,k}w_{ij}^\delta \leq \frac{\theta}{2}|D^2w^\delta|^2 + \frac{C}{2\theta}|Dw^\delta|^2.$$

Then

$$\theta|D^2w^\delta|^2 \leq C(1+|\bar{p}|)|Dw^\delta|^2 + C|Dw^\delta|^3 + \frac{\theta}{2}|D^2w^\delta|^2 + C|\bar{p}|^2|Dw^\delta| \quad \text{at } y_0$$

and $C > 0$ depends only on the L^∞ norm of σ, b, τ and on the derivatives of σ, b and τ. Therefore

$$|D^2w^\delta|^2 \leq C(1+|Dw^\delta|^2 + |\bar{p}||Dw^\delta|^2 + |\bar{p}|^2|Dw^\delta|^2 + |Dw^\delta|^3) \quad \text{at } y_0. \tag{1.23}$$

Thanks to the uniform ellipticity of τ and using equation (1.17), we have

$$\theta|Dw^\delta|^2 \leq |\tau^T Dw^\delta|^2 = \delta w^\delta - \mathrm{tr}(\tau\tau^T D^2w^\delta) - 2\tau\sigma^T\bar{p}\cdot Dw^\delta - b\cdot Dw^\delta \quad \text{at } y_0.$$

Using (1.20), we get at y_0

$$\begin{aligned} z^2 = |Dw^\delta|^4 \leq C(&|\bar{p}|^4 + |D^2w^\delta|^2 + |\bar{p}|^2|Dw^\delta|^2 + |Dw^\delta|^2 + |\bar{p}||Dw^\delta|^2 + |\bar{p}|^2|D^2w^\delta| \\ &+ |\bar{p}|^2|Dw^\delta| + |\bar{p}|^3|Dw^\delta| + |D^2w^\delta||\bar{p}||Dw^\delta| + |D^2w^\delta||Dw^\delta|). \end{aligned} \tag{1.24}$$

Then (1.19) follows by dividing (1.24) by $|Dw^\delta|^3$ and noticing that the right member in (1.24) is polynomial of degree 4 in $|\bar{p}|$ and $|Dw^\delta|$. □

Now we prove Proposition 1.2.1.

Proof of Proposition 1.2.1. We use the methods of [9] based on the small discount approximation (1.17), where F is defined in (1.18). Let $w_\delta := w_\delta(y,\bar{x},\bar{p})$ be the unique periodic continuous solution of (1.17). The regularity theory for viscosity solutions of convex uniformly elliptic equations (see [153] and [141]) gives that $w_\delta \in C^{2,\alpha}$ for some $0<\alpha<1$. Using the Lipschitz estimates proved in Lemma 1.2.2 and the equiboundedness of δw_δ given by (1.20), we obtain that $\delta w_\delta(y)$ converges along a subsequence of $\delta \to 0$ to the constant $\bar{H}(\bar{x},\bar{p})$ and $v_\delta(y) := w_\delta(y) - w_\delta(0)$ converges to the corrector w.

Then, from (1.17) we get

$$\delta v_\delta + \delta w_\delta(0) + F(\bar{x},y,\bar{p},D_y v_\delta, D_{yy}^2 v_\delta) - |\sigma^T(\bar{x},y)\bar{p}|^2 = 0, \quad \text{in } \mathbb{R}^m.$$

Since v_δ is equibounded, $\delta v_\delta \to 0$. Then, since $\delta w_\delta \to \bar{H}$ we get that w is a solution of (1.15). Finally, by the comparison principle for (1.17), it is standard to see that $\bar{H}$ is unique.

Again by the regularity theory (see [153] and [141]), $w \in C^{2,\alpha}$ for some $0 < \alpha < 1$.

Finally the corrector w inherits the Lipschitz estimate of v_δ and satisfies for some $C > 0$ independent of $\bar{p}$ and for all $\bar{x}, \bar{p} \in \mathbb{R}^n$

$$\max_{y \in \mathbb{R}^m} |D_y w(y; \bar{x}, \bar{p})| \leq C(1 + |\bar{p}|).$$

□

1.2.2 Properties and formulas for the effective Hamiltonian and comparison principle

The next result lists some elementary properties of the effective Hamiltonian $\bar{H}$.

Proposition 1.2.3. *(a) $\bar{H}$ is continuous on $\mathbb{R}^n \times \mathbb{R}^n$;*

(b) the function $p \to \bar{H}(x, p)$ is convex;

(c)

$$\min_{y \in \mathbb{R}^m} |\sigma^T(\bar{x}, y)\bar{p}|^2 \leq \bar{H}(\bar{x}, \bar{p}) \leq \max_{y \in \mathbb{R}^m} |\sigma^T(\bar{x}, y)\bar{p}|^2; \tag{1.25}$$

(d) For all $0 < \mu < 1$ and $x, z, q, p \in \mathbb{R}^n$, it holds

$$\mu \bar{H}\left(x, \frac{p}{\mu}\right) - \bar{H}(z, q) \geq \frac{1}{\mu - 1} \sup_{y \in \mathbb{R}^m} |\sigma^T(x, y)p - \sigma^T(z, y)q|^2. \tag{1.26}$$

Proof. The results are obtained by standard methods in the theory of homogenisation, by means of comparison principles for the approximating equation (1.17), see, e.g., [76, 1]. We give the details only on the proof of (1.26). We will use the following inequality:

$$\frac{1}{\mu}|p|^2 - |q|^2 \geq -\frac{1}{1-\mu}|p - q|^2 \quad 0 < \mu < 1, p, q \in \mathbb{R}^n. \tag{1.27}$$

We take w_δ solution of

$$\delta w_\delta + F(z, y, q, D_y w_\delta, D^2_{yy} w_\delta) - |\sigma^T(z, y)q|^2 = 0 \tag{1.28}$$

where F is defined in (1.18), and w^μ_δ solution to

$$\delta w_\delta + F(x, y, \frac{p}{\mu}, D_y w_\delta, D^2_{yy} w_\delta) - |\sigma^T(x, y)\frac{p}{\mu}|^2 = 0. \tag{1.29}$$

We write (1.28) as follows

$$\delta w_\delta - b(y) \cdot D w_\delta - \mathrm{tr}(\tau\tau^T(y) D^2 w_\delta) - |\tau^T(y) D w_\delta + \sigma^T(z, y)q|^2 = 0 \tag{1.30}$$

and similarly we write (1.29) as follows

$$\delta w_\delta^\mu - b(y)\cdot Dw_\delta^\mu - \mathrm{tr}(\tau\tau^T(y)D^2 w_\delta^\mu) - |\tau^T(y)Dw_\delta^\mu + \sigma^T(x,y)\frac{p}{\mu}|^2 = 0.$$

Note that $\mu w_\delta^\mu = \bar{w}$ satisfies

$$\delta\bar{w} - b(y)\cdot D\bar{w} - \mathrm{tr}(\tau\tau^T(y)D^2\bar{w}) - \frac{1}{\mu}|\tau^T(y)D\bar{w} + \sigma^T(x,y)p|^2 = 0. \tag{1.31}$$

By (1.27) we have

$$\begin{aligned}\frac{1}{\mu}|\tau^T(y)Dw_\delta + \sigma^T(x,y)p|^2 \quad &- \quad |\tau^T(y)Dw_\delta + \sigma^T(z,y)q|^2 \\ &\geq \quad -\frac{1}{1-\mu}|\sigma^T(x,y)p - \sigma^T(z,y)q|^2.\end{aligned}$$

Coupling the previous inequality with (1.30) we have

$$\begin{aligned}\delta w_\delta - b(y)\cdot Dw_\delta - \mathrm{tr}(\tau\tau^T(y)D^2 w_\delta) \quad &- \quad \frac{1}{\mu}|\tau^T(y)Dw_\delta + \sigma^T(x,y)p|^2 \\ &\leq \quad \frac{1}{1-\mu}|\sigma^T(x,y)p - \sigma^T(z,y)q|^2.\end{aligned}$$

Since $\bar{w}$ satisfies (1.31), by the comparison principle, we have

$$w_\delta \leq \bar{w} + \frac{1}{1-\mu}\sup_{y\in\mathbb{R}^m}|\sigma^T(x,y)p - \sigma^T(z,y)q|^2 \quad \forall y\in\mathbb{R}^m,$$

that is,

$$w_\delta \leq \mu w_\delta^\mu + \frac{1}{1-\mu}\sup_{y\in\mathbb{R}^m}|\sigma^T(x,y)p - \sigma^T(z,y)q|^2 \quad \forall y\in\mathbb{R}^m$$

from which we easily get (1.26). □

Next we give some representation formulas for the effective Hamiltonian $\bar{H}$.

Proposition 1.2.4. *(i) $\bar{H}$ satisfies*

$$\bar{H}(\bar{x},\bar{p}) = \lim_{\delta\to 0}\sup_{\beta(\cdot)}\delta E\left[\int_0^\infty \left(|\sigma(\bar{x},Z_t)^T\bar{p}|^2 - |\beta(t)|^2\right)e^{-\delta t}dt \,|\, Z_0 = z\right] \tag{1.32}$$

and

$$\bar{H}(\bar{x},\bar{p}) = \lim_{t\to+\infty}\sup_{\beta(\cdot)}\frac{1}{t}E\left[\int_0^t (|\sigma^T(\bar{x},Z_s)\bar{p}|^2 - |\beta(s)|^2)ds \,|\, Z_0 = z\right], \tag{1.33}$$

where $\beta(\cdot)$ is an admissible control process taking values in $\mathbb{R}^r$ for the stochastic control system

$$dZ_t = \left(b(Z_t) + 2\tau(Z_t)\sigma^T(\bar{x},Z_t)\bar{p} - 2\tau(Z_t)\beta(t)\right)dt + \sqrt{2}\tau(Z_t)dW_t; \tag{1.34}$$

(ii) moreover

$$\bar{H}(\bar{x},\bar{p}) = \int_{\mathbb{T}^m} \left(|\sigma(\bar{x},z)^T\bar{p}|^2 - |\tau(z)^T Dw(z)|^2\right) d\mu(z), \tag{1.35}$$

where $w = w(\cdot;\bar{x},\bar{p})$ is the corrector defined in Proposition 1.2.1, $\mathbb{T}^m$ si the m-dimensional torus and $\mu = \mu(\cdot;\bar{x},\bar{p})$ is the invariant probability measure of the process (1.41) *with the feedback $\beta(z) = -\tau^T(z)Dw(z)$;*
(iii) finally

$$\bar{H}(\bar{x},\bar{p}) = \lim_{t\to+\infty} \frac{1}{t}\log E\left[e^{\int_0^t |\sigma^T(\bar{x},Y_s)\bar{p}|^2\,ds}\,|\,Y_0 = y\right], \tag{1.36}$$

where Y_t is the stochastic process defined by

$$dY_t = \left(b(Y_t) + 2\tau(Y_t)\sigma^T(\bar{x},Y_t)\bar{p}\right)dt + \sqrt{2}\tau(Y_t)dW_t. \tag{1.37}$$

Proof. (i) The first formula comes from a control interpretation of the approximating δ-cell problem (1.17). We write it as the Hamilton-Jacobi-Bellman equation

$$\delta w_\delta + \inf_{\beta\in\mathbb{R}^r}\left\{-\mathrm{tr}(\tau(y)\tau(y)^T D^2 w_\delta + \left(2\tau(y)\beta - 2\tau(y)\sigma(\bar{x},y)^T\bar{p} - b(y)\right)\cdot D_y w_\delta + |\beta|^2\right\} - |\sigma(\bar{x},y)^T\bar{p}|^2 = 0 \tag{1.38}$$

and we represent w_δ as the value function of the infinite horizon discounted stochastic control problem (see, e.g., [85])

$$w_\delta(z) = \sup_{\beta(\cdot)} E\left[\int_0^\infty (|\sigma^T(\bar{x},Z_t)\bar{p}|^2 - |\beta(t)|^2)e^{-\delta t}dt\,|\,Z_0 = z\right],$$

where Z_t is defined by (1.34). Then (1.32) follows from the proof of Proposition 1.2.1.

For the formula (1.33) we consider the *t-cell problem*

$$\begin{cases} \frac{\partial v}{\partial t} - \mathrm{tr}(\tau\tau^T D^2 v) - |\tau^T Dv|^2 - (b + 2\tau\sigma^T\bar{p})\cdot Dv - |\sigma^T\bar{p}|^2 = 0 & \text{in } (0,+\infty)\times\mathbb{R}^m, \\ v(0,z) = 0 & \text{on } \mathbb{R}^m. \end{cases} \tag{1.39}$$

This is also a HJB equation, whose solution is the value function

$$v(t,z;\bar{x},\bar{p}) = \sup_{\beta(\cdot)} E\left[\int_0^t (|\sigma^T(\bar{x},Z_s)\bar{p}|^2 - |\beta(s)|^2)ds\,|\,Z_0 = z\right],$$

where Z_t is defined by (1.34). Then a generalized Abelian-Tauberian theorem (see Appendix B, Theorem 2.7.17) states that

$$\bar{H}(\bar{x},\bar{p}) = \lim_{t\to+\infty} \frac{v(t,z;\bar{x},\bar{p})}{t} \quad \text{uniformly in } z. \tag{1.40}$$

(ii) The formula (1.35) is derived from a direct control interpretation of the cell problem (1.15). In fact, it is the HJB equation of the ergodic control problem of maximizing

$$\lim_{T\to\infty} \frac{1}{T} E\left[\int_0^T (|\sigma^T(\bar{x},Z_s)\bar{p}|^2 - |\beta(s)|^2)ds \,|\, Z_0 = z\right],$$

among admissible controls $\beta(\cdot)$ taking values in $\mathbb{R}^r$ for the system (1.34), as before. The process Z_t associated to each control is ergodic with a unique invariant measure μ on the m-dimensional torus $\mathbb{T}^m$ because it is a nondegenerate diffusion on $\mathbb{T}^m$, see, e.g., [3], so the limit in the payoff functional exists and it is the space average in μ of the running payoff (see Proposition 2.7.14 of the Appendix B). Since the HJB PDE (1.15) has a smooth solution w, it is known from a classical verification theorem that the feedback control that achieves the minimum in the Hamiltonian, i.e., $\beta(z) = -\tau^T(z)Dw(z)$, is optimal. Then (1.35) holds with μ the invariant measure of the process

$$d\tilde{Z}_t = \left(b(\tilde{Z}_t) + 2\tau(\tilde{Z}_t)\sigma^T(\bar{x},\tilde{Z}_t)\bar{p} + 2\tau(\tilde{Z}_t)\tau^T(\tilde{Z}_t)Dw(\tilde{Z}_t)\right)dt + \sqrt{2}\tau(\tilde{Z}_t)dW_t. \tag{1.41}$$

(iii) To prove (1.36), take $v = v(t,x;\bar{x},\bar{p})$ a periodic solution of the t-cell problem and define the function $f(t,y) = e^{v(t,y)}$. Then f solves the following equation

$$\begin{cases} \frac{\partial f}{\partial t} - f|\sigma^T\bar{p}|^2 - (2\tau\sigma^T\bar{p} + b)\cdot Df - \mathrm{tr}(\tau\tau^T D^2 f) = 0 & \text{in } (0,\infty)\times\mathbb{R}^m \\ f(0,z) = 1 & \text{in } \mathbb{R}^m. \end{cases}$$

By the Feynman-Kac formula, we have

$$f(t,y) = E\left[e^{\int_0^t |\sigma^T(\bar{x},Y_s)\bar{p}|^2 ds} \,|\, Y_0 = y\right],$$

where Y_t is defined by (1.37). Then

$$v(t,y) = \log E\left[e^{\int_0^t |\sigma^T(\bar{x},Y_s)\bar{p}|^2 ds} \,|\, Y_0 = y\right]$$

and thanks to (1.40), we get (1.36). $\square$

Remark 1.2.5. For $x,p\in\mathbb{R}^n$ define the following perturbed generator $L^{x,p}$

$$L^{x,p}g(y) := Lg(y) + 2(\tau\sigma(x,y)^T p)\cdot D_y g(y),$$

where

$$L = b \cdot D_y + \mathrm{tr}(\tau\tau^T D_{yy}^2).$$

Then the equation (1.15) becomes

$$\bar{H} - e^{-w} L^{\bar{x},\bar{p}} e^w - |\sigma^T \bar{p}|^2 = 0, \tag{1.42}$$

because $e^{-w} L e^w = Lw + |\tau^T D_y w|^2$ gives

$$e^{-w} L^{\bar{x},\bar{p}} e^w = e^{-w} L e^w + 2(\tau\sigma^T \bar{p}) \cdot D_y w = Lw + |\tau^T D_y w|^2 + 2(\tau\sigma^T \bar{p}) \cdot D_y w.$$

Multiplying (1.42) by e^w we get, for $g(y) = e^{w(y)}$,

$$\bar{H} g(y) - (L^{\bar{x},\bar{p}} + V^{\bar{x},\bar{p}}) g(y) = 0, \tag{1.43}$$

where $V^{\bar{x},\bar{p}}(y) = |\sigma^T(\bar{x},y)\bar{p}|^2$ is a multiplicative potential operator.

We conclude that if w is a solution of (1.15), then $\bar{H}$ *is the first eigenvalue of the linear operator* $L^{\bar{x},\bar{p}} + V^{\bar{x},\bar{p}}$*, with eigenfunction* $g = e^w$.

The comparison theorem among viscosity sub- and supersolutions of the limit PDE

$$v_t - \bar{H}(x, Dv) = 0 \quad \text{in } (0,T) \times \mathbb{R}^n \tag{1.44}$$

will be the crucial tool for proving that the convergence of v^ε is not only in the weak sense of semilimits but in fact uniform, and the limit is unique. We observe that property (d) of Proposition 1.2.3 is crucial in the proof since it allows us to relate the regularity in x of $\bar{H}$ with that of the pseudo-coercive Hamiltonian $|\sigma^T(x,y)p|^2$. With this inequality one can repeat the proof of the comparison principle for the pseudo-coercive Hamiltonian by Barles and Perthame (see [31] for the stationary case and [11] for the evolutionary case).

Theorem 1.2.6. *Let* $u \in BUSC([0,T] \times \mathbb{R}^n)$ *and* $v \in BLSC([0,T] \times \mathbb{R}^n)$ *be, respectively, a bounded upper semicontinuous subsolution and a bounded lower semicontinuous supersolution to*

$$v_t = \bar{H}(x, Dv) \quad \text{in } (0,T) \times \mathbb{R}^n$$

such that $u(0,x) \le h(x) \le v(0,x)$ *for all* $x \in \mathbb{R}^n$*. Then* $u(x,t) \le v(x,t)$ *for all* $x \in \mathbb{R}^n$ *and* $0 \le t \le T$*.*

Proof. Let us give a sketch of the main points of the proof. We show that for $\mu < 1$, μ sufficiently near to 1, it holds

$$\sup_{\mathbb{R}^n \times [0,T]} (u - \mu v) \le \sup_{\mathbb{R}^n} (u - \mu v)(\cdot, 0).$$

If this is true, then the inequality holds also for $\mu = 1$, proving the theorem. By contradiction, we assume that for every $\mu < 1$, there exists $(\bar{x},\bar{t})$ such that

$$u(\bar{x},\bar{t}) - \mu v(\bar{x},\bar{t}) > \sup_{\mathbb{R}^n}(u - \mu v)(\cdot,0). \tag{1.45}$$

Let

$$\Phi(x,z,t,s) = u(x,t) - \mu v(z,s) - \frac{|x-z|^2}{\varepsilon^2} - \frac{|t-s|^2}{\eta^2} - \delta \log(1 + |x|^2 + |z^2|) + \alpha\mu s.$$

For ε, η small enough, Φ has a maximum point, that we denote with (x', z', t', s'). By standard arguments, we get $\frac{|x'-z'|^2}{\varepsilon^2}, \frac{|t'-s'|^2}{\eta^2} \longrightarrow 0$ as $\varepsilon, \eta \to 0$.

If either $s' = 0$ or $t' = 0$, it is easy to see that we get a contradiction with (1.45). So we consider the case $(x', z', t', s') \in \mathbb{R}^n \times \mathbb{R}^n \times (0,T) \times (0,T)$. Let

$$p = 2\frac{x'-z'}{\varepsilon^2}, \quad q_x = \frac{2x'}{1+|x'|^2+|z'|^2}, \quad q_z = \frac{-2y'}{1+|x'|^2+|z'|^2}, \quad r = 2\frac{t'-s'}{\eta^2}.$$

Using the fact that u is a subsolution we get

$$r - \bar{H}(x', p + \delta q_x) \leq 0. \tag{1.46}$$

Since v is a supersolution, we get

$$\frac{r}{\mu} - \bar{H}\left(z', \frac{p + \delta q_z}{\mu}\right) \geq \alpha \tag{1.47}$$

So, we multiply (1.47) by $-\mu$ and sum up to (1.46) to obtain

$$\mu \bar{H}\left(z', \frac{p + \delta q_z}{\mu}\right) - \bar{H}(x', p + \delta q_x) \leq -\alpha\mu. \tag{1.48}$$

Using Proposition 1.2.3, property (d), we get

$$\begin{aligned} &\mu \bar{H}\left(z', \frac{p + \delta q_z}{\mu}\right) - \bar{H}(x', p + \delta q_x) \\ &\qquad \geq -\frac{1}{1-\mu} \sup_{y \in \mathbb{R}^m} |\sigma^T(z',y)(p + \delta q_z) - \sigma^T(x',y)(p + \delta q_x)|^2. \end{aligned} \tag{1.49}$$

Now we prove that

$$|\sigma^T(z',y)(p + \delta q_z) - \sigma^T(x',y)(p + \delta q_x)|^2 \tag{1.50}$$

goes to zero uniformly in y as $\varepsilon, \eta, \delta$ go to zero, reaching a contradiction. In fact we write (1.50) as

$$|\sigma^T(z',y)(p+\delta q_z) - \sigma^T(x',y)(p+\delta q_x)|^2 \\ = |((\sigma(x',y) - \sigma(z',y))^T(p+\delta q_z) + \delta\sigma^T(x',y)(q_x - q_z)|^2.$$

Let

$$\Delta(y) = ((\sigma(x',y) - \sigma(z',y))^T(p+\delta q_z), \quad J(y) = \delta\sigma^T(x',y)(q_x - q_z).$$

Then, since $\Delta(y)$ goes to zero for $\varepsilon, \eta \to 0$ and for all δ fixed uniformly in y, and $J(y)$ goes to zero for $\varepsilon, \eta, \delta \to 0$ uniformly in y, we conclude that the limit in the right-hand side of (1.49) is zero, reaching a contradiction. □

1.3 The supercritical case: $\alpha > 2$

As in Section 1.2, we prove the existence of an effective Hamiltonian giving the limit PDE and first we identify the cell problem that we wish to solve. Plugging the asymptotic expansion

$$v^\varepsilon(t,x,y) = v^0(t,x) + \varepsilon^{\alpha-1}w(t,x,y)$$

in the equation (1.10) satisfied by v^ε we get

$$v^0_t = |\sigma^T D_x v^0|^2 + b \cdot D_y w + \mathrm{tr}(\tau\tau^T D^2_{yy} w) + O(\varepsilon).$$

Then the true cell problem is to find the unique constant $\bar{H}(\bar{x},\bar{p})$ such that

$$\bar{H}(\bar{x},\bar{p}) - |\sigma(\bar{x},y)^T\bar{p}|^2 - b(y)\cdot D_y w(y) - \mathrm{tr}(\tau(y)\tau(y)^T D^2_{yy} w = 0 \text{ in } \mathbb{R}^m, \tag{1.51}$$

has a periodic viscosity solution w. We consider the δ-*cell problem* for fixed $(\bar{x},\bar{p},\bar{X})$

$$\delta w_\delta(y) - |\sigma(\bar{x},y)^T\bar{p}|^2 - b(y)\cdot D_y w_\delta(y) - \mathrm{tr}(\tau(y)\tau(y)^T D^2_{yy} w_\delta(y)) = 0 \text{ in } \mathbb{R}^m, \tag{1.52}$$

where w_δ is the *approximate corrector*. The next result states that δw_δ converges to $\bar{H}$ and it is smooth.

Proposition 1.3.1. *For any fixed $(\bar{x},\bar{p})$ there exists a constant $\bar{H}(\bar{x},\bar{p})$ such that $\bar{H}(\bar{x},\bar{p}) = \lim_{\delta\to 0}\delta w_\delta(y)$ uniformly, where $w_\delta \in C^2(\mathbb{R}^m)$ is the unique periodic solution of* (1.52). *Moreover*

$$\bar{H}(\bar{x},\bar{p}) := \int_{\mathbb{T}^m} |\sigma(\bar{x},y)^T\bar{p}|^2 \, d\mu(y), \tag{1.53}$$

where μ is the invariant probability measure on the m-dimensional torus $\mathbb{T}^m$ of the stochastic process

$$dY_t = b(Y_t)dt + \sqrt{2}\tau(Y_t)dW_t,$$

that is, the periodic solution of

$$-\sum_{i,j} \frac{\partial^2}{\partial y_i \partial y_j}((\tau\tau^T)_{ij}(y))\mu + \sum_i \frac{\partial}{\partial y_i}(b_i(y))\mu = 0 \qquad in\ \mathbb{R}^m, \tag{1.54}$$

with $\int_{\mathbb{T}^n} \mu(y)\,dy = 1$.

Proof. The proof essentially follows the arguments presented in [9, 3] of ergodic control theory in periodic enviroments. We just notice that the process Y_t is ergodic since it is a uniformly elliptic diffusion on the torus (see [3]) and then it has a unique invariant probability measure μ (see the Appendix B for definitions of ergodicity and invariant measure). We finally observe that (1.53) is a necessary and sufficient condition for the true cell problem to have a solution. This is a known result that follows formally from multiplying (1.51) by μ and integrating by parts and using (1.54). $\square$

Remark 1.3.2. Note that in dimension $n = 1$ the effective Hamiltonian assumes the form

$$H(\bar{x}, \bar{p}) = \int_{\mathbb{T}^m} \sigma(\bar{x}, y)^2 d\mu(y)\bar{p}^2 = (\bar{\sigma}\bar{p})^2,$$

where $\bar{\sigma} = \sqrt{\int_{\mathbb{T}^m} \sigma(\bar{x}, y)^2 d\mu(y)}$.

We observe that the effective Hamiltonian $\bar{H}$ satisfies properties $(a), (b), (c), (d)$ as in Proposition 1.2.3, which can be proven with similar arguments. Then, the proof of Theorem 1.2.6 applies here and we have the following comparison result among viscosity sub- and supersolutions of the limit PDE

$$v_t - \int_{\mathbb{T}^m} |\sigma(x,y)^T Dv|^2\, d\mu(y) = 0 \quad \text{in } (0,T) \times \mathbb{R}^n. \tag{1.55}$$

Theorem 1.3.3. *Let $u \in BUSC([0,T] \times \mathbb{R}^n)$ and $v \in BLSC([0,T] \times \mathbb{R}^n)$ be, respectively, a subsolution and a supersolution to* (1.55) *such that $u(0,x) \le v(0,x)$ for all $x \in \mathbb{R}^n$. Then $u(x,t) \le v(x,t)$ for all $x \in \mathbb{R}^n$ and $0 \le t \le T$.*

1.4 The subcritical case: $\alpha < 2$

In this case, the asymptotic expansion we plug in the equation is

$$v^{\varepsilon}(t,x,y) = v^0(t,x) + \varepsilon^{\frac{\alpha}{2}} w(t,x,y). \tag{1.56}$$

Plugging (1.56) into the equation (1.10) satisfied by v^{ε}, we get

$$v^0_t = |\sigma^T D_x v^0|^2 + 2(\tau\sigma^T D_x v^0)\cdot D_y w + |\tau^T D_y w|^2 + O(\varepsilon). \tag{1.57}$$

Therefore the cell problem we want to solve is finding, for any fixed $(\bar{x},\bar{p})$, a unique constant $\bar{H}$ such that there is a viscosity solution w of the following equation

$$\bar{H}(\bar{x},\bar{p}) - 2(\tau(y)\sigma(\bar{x},y)^T\bar{p})\cdot D_y w(y) - |\tau(y)^T D_y w(y)|^2 - |\sigma(\bar{x},y)^T\bar{p}|^2 = 0. \tag{1.58}$$

Since

$$2(\tau(y)\sigma^T(\bar{x},y)\bar{p})\cdot D_y w = 2(\sigma^T(\bar{x},y)\bar{p})\cdot(\tau^T(y)D_y w),$$

we can restate the cell problem as

$$\bar{H}(\bar{x},\bar{p}) - |\tau^T(y)D_y w(y) + \sigma^T(\bar{x},y)\bar{p}|^2 = 0. \tag{1.59}$$

The following proposition deals with the existence and uniqueness of $\bar{H}$.

Proposition 1.4.1. *For any fixed $(\bar{x},\bar{p})$, there exists a unique constant $\bar{H}(\bar{x},\bar{p})$ such that the cell problem* (1.58) *admits a periodic viscosity solution w. Moreover w is Lipschitz continuous and there exists $C > 0$ independent of $\bar{x},\bar{p}$ such that*

$$\max_y |Dw(y;\bar{x},\bar{p})| \le C(1+|\bar{p}|).$$

Proof. As for the other cases we introduce the following approximant problem, with $\delta > 0$,

$$\delta w_\delta(y) - |\tau^T(y)D_y w_\delta(y) + \sigma^T(\bar{x},y)\bar{p}|^2 = 0 \text{ in } \mathbb{R}^m. \tag{1.60}$$

Let w_δ the unique periodic viscosity solution to (1.60). By standard comparison principle we get that

$$|\delta w_\delta| \le \max_{y\in\mathbb{R}^m} |\sigma^T(\bar{x},y)\bar{p}|^2 \le C(1+|\bar{p}|^2) \quad \forall y\in\mathbb{R}^m.$$

Moreover, using the coercivity of the Hamiltonian (see [14, Prop II.4.1]), we get that w_δ is Lipschitz continuous and there exists a constant C independent of δ and $\bar{p}$ such that

$$\max_{y\in\mathbb{R}^m} |Dw_\delta| \le C(1+|\bar{p}|).$$

So, we conclude as in the proof of Proposition 1.2.1. □

We give some representation formulas for the effective Hamiltonian $\bar{H}$.

Proposition 1.4.2. *(i) $\bar{H}$ satisfies*

$$\bar{H}(\bar{x},\bar{p}) = \lim_{\delta\to 0}\sup_{\beta(\cdot)} \delta \int_0^{+\infty} \left(|\sigma(\bar{x},y(t))^T\bar{p}|^2 - |\beta(t)|^2\right) e^{-\delta t}\,dt, \tag{1.61}$$

where $\beta(\cdot)$ varies over measurable functions taking values in $\mathbb{R}^r$, $y(\cdot)$ is the trajectory of the control system

$$\begin{cases} y'(t) = 2\tau(y(t))\sigma^T(\bar{x},y(t))\bar{p} - 2\tau(y(t))\beta, & t>0,\\ y(0)=y \end{cases}$$

and the limit is uniform with respect to the initial position y of the system;
(ii) if, in addition, $\tau(y)\sigma^T(x,y)=0$ for all x,y, then

$$\bar{H}(\bar{x},\bar{p}) = \max_{y\in\mathbb{R}^m} |\sigma^T(\bar{x},y)\bar{p}|^2; \tag{1.62}$$

(iii) if $n=m=r=1$, and $\sigma\geq 0$

$$\bar{H}(\bar{x},\bar{p}) = \left(\int_0^1 \frac{\sigma(\bar{x},y)}{\tau(y)}\,dy\right)^2 \left(\int_0^1 \frac{1}{\tau(y)}\,dy\right)^{-2} \bar{p}^2. \tag{1.63}$$

Proof. The formula (1.61) can be proved by writing (1.60) as a Bellman equation

$$\delta w_\delta(y) + \inf_{\beta\in\mathbb{R}^r}\left\{\left(2\tau(y)\beta - 2\tau(y)\sigma(\bar{x},y)^T\bar{p}\right)\cdot D_y w_\delta + |\beta|^2\right\} - |\sigma(\bar{x},y)^T\bar{p}|^2 = 0. \tag{1.64}$$

Then w_δ is the value function of the infinite horizon discounted deterministic control problem appearing in (1.61) (see, e.g., [14, 22]). If $\tau(y)\sigma^T(x,y)=0$ for all x,y, then (1.59) reads

$$-|\tau^T(y)D_y w(y)|^2 = |\sigma^T(\bar{x},y)\bar{p}|^2 - \bar{H}(\bar{x},\bar{p}).$$

So, this gives immediately the inequality $\geq$ in (1.62). The other inequality is obtained by standard comparison principle arguments applied to the approximating problem (1.60).
Finally, in the case $n=m=r=1$, if $\bar{p}\geq 0$ we write explicitly the corrector as

$$w(y) = \int_0^y \frac{\bar{H}^{\frac{1}{2}} - \sigma(\bar{x},s)\bar{p}}{\tau(s)}\,ds.$$

Note that $w\in C^1$ is periodic and does the job. A similar construction works for $\bar{p}<0$. □

We observe that $\bar{H}$ satisfies properties (a), (b), (c), (d) as in Proposition 1.2.3, which can be proven with similar arguments. Then, the proof of Theorem 1.2.6 applies here and we have the following comparison result among viscosity sub- and supersolutions of the limit PDE

$$v_t - \bar{H}(x, Dv) = 0 \quad \text{in } (0,T)\times\mathbb{R}^n. \tag{1.65}$$

Theorem 1.4.3. *Let $u \in BUSC([0,T]\times\mathbb{R}^n)$ and $v \in BLSC([0,T]\times\mathbb{R}^n)$ be, respectively, a bounded upper semicontinuous subsolution and a bounded lower semicontinuous supersolution to* (1.65) *such that $u(0,x) \le h(x) \le v(0,x)$ for all $x \in \mathbb{R}^n$. Then $u(x,t) \le v(x,t)$ for all $x \in \mathbb{R}^n$ and $0 \le t \le T$.*

1.5 The convergence result

In this section we state the convergence theorem for the singular perturbation problem. We will make use of the relaxed semi-limits which we define as follows. For the functions v_ε defined in (1.9) the relaxed upper semi-limit $\bar{v} = \limsup^*_{\varepsilon\to 0}\sup_y v^\varepsilon$ is

$$\bar{v}(t,x) := \limsup_{\varepsilon\to 0,(t',x')\to(t,x)} \sup_y v^\varepsilon(t',x',y), \quad x\in\mathbb{R}^n, t\ge 0.$$

We define analogously the lower semi-limit $\underline{v} = \liminf_{*\varepsilon\to 0}\inf_y v^\varepsilon$ by replacing $\limsup$ with $\liminf$ and sup with inf. Since h is bounded the family v^ε is equibounded and we have $\bar{v} \in BUSC([0,T]\times\mathbb{R}^n)$ and $\underline{v} \in BLSC([0,T]\times\mathbb{R}^n)$.

First we state the convergence result in the critical and supercritical case $\alpha \ge 2$.

Recall that by Proposition 1.1.2 i) v^ε defined by (1.9) is the solution of

$$\begin{cases} \partial_t v^\varepsilon - H^\varepsilon\left(x, y, D_x v^\varepsilon, \frac{D_y v^\varepsilon}{\varepsilon^{\alpha-1}}, D_{xx} v^\varepsilon, \frac{D^2_{yy} v^\varepsilon}{\varepsilon^{\alpha-1}}, \frac{D_{xy} v^\varepsilon}{\varepsilon^{\frac{\alpha-1}{2}}}\right) = 0 & (0,T)\times\mathbb{R}^n\times\mathbb{R}^m \\ v^\varepsilon(0,x,y) = h(x) & \mathbb{R}^n\times\mathbb{R}^m. \end{cases}$$

with

$$\begin{aligned} H^\varepsilon(x,y,p,q,X,Y,Z) : &= |\sigma^T p|^2 + b\cdot q + \mathrm{tr}(\tau\tau^T Y) + \varepsilon\left(\mathrm{tr}(\sigma\sigma^T X) + \phi\cdot p\right) \\ &+ 2\varepsilon^{\frac{\alpha}{2}-1}(\tau\sigma^T p)\cdot q + 2\varepsilon^{\frac{1}{2}}\mathrm{tr}(\sigma\tau^T Z) + \varepsilon^{\alpha-2}|\tau^T q|^2. \end{aligned}$$

Theorem 1.5.1. *Assume $\alpha \ge 2$. Then*

i) The upper limit $\bar{v}$ (resp., the lower limit $\underline{v}$) of v^ε is a subsolution (resp., supersolution) of the effective equation

$$v_t - \bar{H}(x,Dv) = 0 \text{ in } (0,T)\times\mathbb{R}^n \quad v(0,x) = h(x) \text{ on } \mathbb{R}^n \tag{1.66}$$

where $\bar{H}$ is given by (1.53) *for $\alpha > 2$, and it is defined by Proposition 1.2.1 for $\alpha = 2$ (with the formulas* (1.32), (1.33), (1.35)*, and* (1.36)*);*
ii) v^ε converges uniformly on the compact subsets of $[0,T) \times \mathbb{R}^n \times \mathbb{R}^m$ *to the unique viscosity solution of* (1.66).

Proof. i) The inequalities $\underline{v}(0,x) \le h(x) \le \bar{v}(0,x)$ follow from the definitions. The problem of taking the limit in the PDE is a regular perturbation of a singular perturbation problem, in the terminology of [4]. The result can be proved by the methods developed in [4] for such problems, with minor modifications.
ii) By the definition of the semilimits $\underline{v} \le \bar{v}$ in $[0,T) \times \mathbb{R}^n$. The comparison principle (Theorem 1.2.6 and Theorem 1.3.3) for the effective equation (1.66) gives the inequality $\le$ and therefore $\bar{v} = \underline{v} = v$ in $[0,T] \times \mathbb{R}^n$. Thanks to the properties of semilimits, we finally get that v^ε converges locally uniformly to the unique bounded solution of (1.66). □

Now we state the convergence result in the subcritical case, $\alpha < 2$.
Recall that by Proposition 1.1.2 ii) v^ε defined by (1.9) is the solution of

$$\begin{cases} v_t^\varepsilon = H_\varepsilon\left(x, y, D_x v^\varepsilon, \frac{D_y v^\varepsilon}{\varepsilon^{\frac{\alpha}{2}}}, D_{xx} v^\varepsilon, \frac{D_{yy}^2 v^\varepsilon}{\varepsilon^{\frac{\alpha}{2}}}, \frac{D_{xy} v^\varepsilon}{\varepsilon^{\frac{\alpha}{4}}}\right) & (0,T) \times \mathbb{R}^n \times \mathbb{R}^m \\ v^\varepsilon(0,x,y) = h(x) & \mathbb{R}^n \times \mathbb{R}^m. \end{cases}$$

with

$$\begin{aligned} H_\varepsilon(x,y,p,q,X,Y,Z) : &= |\sigma^T p|^2 + 2(\tau\sigma^T p)\cdot q + |\tau^T q|^2 + \varepsilon\left(\mathrm{tr}(\sigma\sigma^T X) + \phi\cdot p\right) \\ &+ 2\varepsilon^{1-\frac{\alpha}{4}} \mathrm{tr}(\sigma\tau^T Z) + \varepsilon^{1-\frac{\alpha}{2}} b\cdot q + \varepsilon^{1-\frac{\alpha}{2}} \mathrm{tr}(\tau\tau^T Y). \end{aligned}$$

Theorem 1.5.2. *Assume $\alpha < 2$. Then*
i) the upper limit $\bar{v}$ (resp., the lower limit $\underline{v}$) of v^ε is a subsolution (resp., supersolution) of the effective equation (1.66) *where $\bar{H}$ is defined by Proposition 1.4.1 (with the formula* (1.61)*);*
ii) v^ε converges uniformly on the compact subsets of $[0,T) \times \mathbb{R}^n \times \mathbb{R}^m$ *to the unique viscosity solution of* (1.66).

Proof. The proof is the same as that of Theorem 1.5.1, by using the comparison principle Theorem 1.4.3. □

Ordering of the three cases

The convergence theorem stated above and the formulas for $\bar{H}$ say that there are three possible limits for v^ε, depending only on the position of α with respect to the critical value $\alpha = 2$. Let us call them v_{sup}, v_c and v_{sub}, if, respectively, $\alpha > 2, \alpha = 2$, or $\alpha < 2$. We can compare them in the uncorrelated case.

Corollary 1.5.3. *If $\tau(y)\sigma^T(x,y) = 0$ for all x,y, then*

$$v_{sup}(t,x) \le v_c(t,x) \le v_{sub}(t,x) \quad \forall t \ge 0, x \in \mathbb{R}^n. \tag{1.67}$$

Proof. If $\tau\sigma^T = 0$ we can easily compare the three effective Hamiltonians $\bar{H}_{sup}, \bar{H}_c$, and $\bar{H}_{sub}$, respectively. In fact, (1.25) and (1.62) give

$$\bar{H}_c(x,p) \le \bar{H}_{sub}(x,p) \quad \forall x \in \mathbb{R}^n, p \in \mathbb{R}^n.$$

On the other hand, using the control $\beta \equiv 0$ in (1.34) and by the assumption $\tau\sigma^T = 0$, we get the diffusion

$$dY_t = b(Y_t)dt + \sqrt{2}\tau(Y_t)dW_t,$$

whose invariant measure μ appears in the formula (1.53) for $\bar{H}_{sup}$. Then (1.33) gives

$$\bar{H}_c(x,p) \ge \lim_{t\to+\infty} \frac{1}{t} E\left[\int_0^t |\sigma^T(x,Y_s)p|^2\, ds\right] = \int_{\mathbb{T}^m} |\sigma^T(x,y)p|^2\, d\mu(y) = \bar{H}_{sup}(x,p)$$

for all initial condition Y_0. Now the inequalities (1.67) are obtained by the comparison principle Theorem 1.2.6. $\square$

1.6 Large deviation and applications to option pricing

The large deviation principle

In this chapter we derive a large deviation principle for the process X_t^ε defined in (1.6). Throughout the section we suppose that σ is uniformly non degenerate, that is, for some $\nu > 0$ and for all $x, p \in \mathbb{R}^n$

$$|\sigma^T(x,y)p|^2 > \nu|p|^2. \tag{1.68}$$

By (1.25), under (1.68), the effective Hamiltonian is coercive. Let $\bar{L}$ be the *effective Lagrangian*, i.e. for $x \in \mathbb{R}^n$

$$\bar{L}(x,q) = \max_{p\in\mathbb{R}^n}\{p\cdot q - \bar{H}(x,p)\}. \tag{1.69}$$

Note that $\bar{L}(x,\cdot)$ is a convex nonnegative function such that $\bar{L}(x,0) = 0$ for all $x \in \mathbb{R}^n$, since $\bar{H}(x,\cdot)$ is convex nonnegative and $\bar{H}(x,0) = 0$ for all $x \in \mathbb{R}$.

For each $x_0 \in \mathbb{R}^n$ and $t > 0$, define

$$I(x;x_0,t) := \inf\left[\int_0^t \bar{L}\left(\xi(s),\xi'(s)\right)\, ds \,\middle|\, \xi \in AC(0,t),\ \xi(0) = x_0, \xi(t) = x\right]. \tag{1.70}$$

Remark 1.6.1. (a) The function I defined in (1.70) is continuous in the variable x (see, e.g., [70]) and is a nonnegative function such that $I(x_0;x_0,t) = 0$.
(b) I satisfies the following growth condition for some $C > 0$ and all $x, x_0 \in \mathbb{R}^n$

$$\frac{1}{4C}\frac{|x-x_0|^2}{t} \le I(x;x_0,t) \le \frac{1}{4\nu}\frac{|x-x_0|^2}{t}, \tag{1.71}$$

where ν is defined in (1.68). In fact, thanks to the property (1.25) stated in Proposition 1.2.3, we get that

$$\frac{1}{4C}|p|^2 \le \bar{L}(x,p) \le \frac{1}{4\nu}|p|^2.$$

Then we have

$$\frac{1}{4C}\inf_{\xi(0)=x_0,\xi(t)=x}\int_0^t |\xi'(s)|^2 \le I(x;x_0,t) \le \frac{1}{4\nu}\inf_{\xi(0)=x_0,\xi(t)=x}\int_0^t |\xi'(s)|^2,$$

from which we get (1.71).
(c) If σ does not depend on x, i.e. $\bar{H} = \bar{H}(p)$, the rate function in (1.70) is

$$I(x;x_0,t) = t\bar{L}\left(\frac{x-x_0}{t}\right).$$

(d) If σ does not depend of x and $n = 1$, I is a monotone nondecreasing function of x when $x > x_0$. Analogously, I is a monotone nonincreasing function of x when $x < x_0$.

Theorem 1.6.2. *Let* $(X^\varepsilon, Y^\varepsilon)$ *be the process defined in* (1.6) *with initial position* $X_0^\varepsilon = x_0$ *and* $Y_0^\varepsilon = y_0$. *Then for every* $t > 0$, *a large deviation principle holds for* $\{X_t^\varepsilon : \varepsilon > 0\}$ *with speed* $\frac{1}{\varepsilon}$ *and good rate function* $I(x;x_0,t)$. *In particular, for any open set* $B \subseteq \mathbb{R}^n$

$$\lim_{\varepsilon\to 0}\varepsilon \log P(X_t^\varepsilon \in B) = -\inf_{x\in B} I(x;x_0,t). \tag{1.72}$$

Remark 1.6.3. Thanks to Remark 1.6.1, if σ does not depend on x and $n = 1$, we have $\inf_{y>x} I(y;x_0,t) = I(x;x_0,t)$ for $x \ge x_0$ and (1.72) can be written in the following way

$$\lim_{\varepsilon\to 0}\varepsilon \log P(X_t^\varepsilon > x) = -I(x;x_0,t) \quad \text{when } x > x_0$$

and analogously when $x < x_0$

$$\lim_{\varepsilon\to 0}\varepsilon \log P(X_t^\varepsilon < x) = -I(x;x_0,t).$$

Remark 1.6.4. We note that the rate function I defined in (1.70) does not depend on the drift ϕ of the log-price X_t^ε and it depends only on the volatility σ and on the fast process Y_t^ε. In fact, this holds for the effective Hamiltonian $\bar{H}$ by the representation formulas (1.32) for

$\alpha = 2$, (1.53) for $\alpha > 2$ and (1.61) for $\alpha < 2$, and hence it holds for the Legendre transform $\bar{L}$.

Proof. We divide the proof in two steps, the first is the proof of the large deviation principle, while the second is the proof of the representation formula (1.70) for the good rate function.

Step. *1 (Large deviation principle)* The proof of this step is similar to that of Theorem 2.1 of [79] with some minor changes. The idea is to apply Bryc's inverse Varadhan lemma (see Appendix A, Lemma 2.7.9) with μ_ε given by the laws of $\{X_t^\varepsilon\}$ and Λ_h^ε given by v_ε. Recall that, for $h \in BC(\mathbb{R}^n)$, v_ε is defined as

$$v_\varepsilon(t,x,y) := \varepsilon \log E\left[e^{\frac{h(X_t^\varepsilon)}{\varepsilon}} |(X_\cdot^\varepsilon, Y_\cdot^\varepsilon) \text{ satisfy (1.6)}\right].$$

We proved in Theorems 1.5.1, 1.5.2 that v_ε converge uniformly to a function v^h.
In order to apply Lemma 2.7.9 (Appendix A), we have to prove the exponential tightness of $\{X_t^\varepsilon\}$. Define the following function

$$f_\varepsilon(x,y) = \begin{cases} f(x) + \varepsilon^{\alpha-1}\zeta(y) & \text{if } \alpha \geq 2, \\ f(x) + \varepsilon^{\frac{\alpha}{2}}\zeta(y) & \text{if } \alpha < 2, \end{cases} \tag{1.73}$$

where

$$f(x) = \log(1+|x|^2)$$

and $\zeta(y)$ is a positive differentiable function with bounded first and second derivatives. Since $f(x)$ is an increasing function of $|x|$ and since $\zeta(y) \geq 0$, we have that for any $c > 0$ there exists a compact set $K_c \subset \mathbb{R}^n$ such that

$$f_\varepsilon(x,y) > c \ \text{ when } x \notin K_c. \tag{1.74}$$

We observe that $||\frac{\partial f}{\partial_{x_j}}||_\infty + ||\frac{\partial^2 f}{\partial_{x_j x_i}}||_\infty < \infty$ for all $i = 1 \cdots n, j = 1 \cdots n$, and by our choice of ζ we therefore have that

$$\sup_{x\in\mathbb{R}^n, y\in\mathbb{R}^m} H_\varepsilon(x,y,D_x f_\varepsilon, D_y f_\varepsilon, D^2_{xx} f_\varepsilon, D^2_{yy} f_\varepsilon, D^2_{xy} f_\varepsilon) = C < \infty, \tag{1.75}$$

where H_ε is defined as follows

$$\begin{aligned} H_\varepsilon(x,y,p,q,X,Y,Z) &= |\sigma^T p|^2 + \varepsilon \mathrm{tr}(\sigma\sigma^T X) + \varepsilon\phi \cdot p + 2\varepsilon^{-\frac{\alpha}{2}} \mathrm{tr}(\tau\sigma^T p)\cdot q \\ &+ 2\varepsilon^{1-\frac{\alpha}{2}} \mathrm{tr}(\sigma\tau^T Z) + \varepsilon^{1-\alpha} b\cdot q + \varepsilon^{-\alpha}|\tau^T q|^2 + \varepsilon^{1-\alpha}\mathrm{tr}(\tau\tau^T Y). \end{aligned}$$

We will write $H_\varepsilon f_\varepsilon(x,y)$ to denote $H_\varepsilon(x,y,D_x f_\varepsilon, D_y f_\varepsilon, D^2_{xx} f_\varepsilon, D^2_{yy} f_\varepsilon, D^2_{xy} f_\varepsilon)$. The P and E in the following proof denote probability and expectation conditioned on (X,Y) starting at

(x,y). Define the process

$$M_t^\varepsilon = \exp\left\{\frac{f_\varepsilon(X_t^\varepsilon, Y_t^\varepsilon)}{\varepsilon} - \frac{f_\varepsilon(x,y)}{\varepsilon} - \frac{1}{\varepsilon}\int_0^t H_\varepsilon f_\varepsilon(X_s^\varepsilon, Y_s^\varepsilon)\,ds\right\}. \tag{1.76}$$

Then $M_{\varepsilon,t}$ is a supermartingale and hence we can apply the optional sampling theorem (see Appendix A, Theorem 2.7.10), that is

$$1 \geq E\,[M_t^\varepsilon]. \tag{1.77}$$

Then

$$\begin{aligned} 1 \geq E\,[M_t^\varepsilon\,|X_t^\varepsilon \notin K_c] &\geq E\left[e^{\frac{(c-f_\varepsilon(x,y)-tC)}{\varepsilon}}\,|X_t^\varepsilon \notin K_c\right] \\ &= P(X_t^\varepsilon \notin K_c)e^{\frac{(c-f_\varepsilon(x,y)-tC)}{\varepsilon}}, \end{aligned} \tag{1.78}$$

where we have used (1.74) and (1.75) to estimate the first and third term in M_t^ε. Then we get

$$\varepsilon \log P(X_t^\varepsilon \notin K_c) \leq tC + f_\varepsilon(x,y) - c \leq \text{ const } - c$$

and this finally gives us the exponential tightness of X_t^ε.
So, by Bryc's inverse Varadhan lemma (see Appendix A, Lemma 2.7.9), the measures associated to the process X_t^ε satisfy the LDP with the good rate function

$$I(x;x_0,t) = \sup_{h\in BC(\mathbb{R}^n)} \{h(x) - v^h(t,x_0)\} \tag{1.79}$$

and

$$v^h(t,x_0) = \sup_{x\in\mathbb{R}^n}\{h(x) - I(x;x_0,t)\}.$$

Step. *2 (Representation formula for the good rate function)* The solution v^h to the effective equation

$$\begin{cases} v_t - \bar{H}(x,Dv) = 0 & \text{in } (0,T)\times\mathbb{R}^n \\ v(0,x) = h(x) & \text{in } \mathbb{R}^n \end{cases} \tag{1.80}$$

can be represented through the following formula

$$v^h(t,x) =$$
$$\sup\left\{h(y) - \int_0^t \bar{L}\left(\xi(s),\xi'(s)\right)\,ds \mid y\in\mathbb{R}^n, \xi\in AC(0,t), \xi(0)=x, \xi(t)=y\right\}, \tag{1.81}$$

where $\bar{L}$ is the effective Lagrangian defined in (1.69). We refer to [70] where it is shown that v^h is continuous and is the solution of (1.80). We define

$$r(x;x_0,t) = \inf_{\xi(0)=x_0,\xi(t)=x} \int_0^t \bar{L}\left(\xi(s),\xi'(s)\right) ds \tag{1.82}$$

Thanks to (1.79) and (1.81), we can write

$$I(x;x_0,t) = \\ r(x;x_0,t) \quad + \quad \sup_{h\in BC(\mathbb{R})} \inf \left\{ h(x) - h(y) + \int_0^t \bar{L}\left(\xi(s),\xi'(s)\right) ds - r(x;x_0,t) \right\}, \tag{1.83}$$

where the infimum is over $y \in \mathbb{R}^n$ and absolutely continuous functions ξ such that $\xi(0) = x_0, \xi(t) = y$. Then

$$I(x;x_0,t) = r(x;x_0,t) + J(x;x_0,t),$$

where $J(x;x_0,t) := \sup_{h\in BC(\mathbb{R})} J_h(x;x_0,t)$ and

$$J_h(x;x_0,t) = \inf \left\{ h(x) - h(y) + \int_0^t \bar{L}\left(\xi(s),\xi'(s)\right) ds - r(x;x_0,t) \right\}.$$

Taking $y = x$, we obtain $J_h(x;x_0,t) \leq 0$ and therefore $J(x;x_0,t) \leq 0$. Now we define a function $h_* \in BC(\mathbb{R})$ as follows:

$$h_*(y) = r(y;x_0,t) \wedge r(x;x_0,t).$$

We claim that h_* is continuous. Then $J_{h_*}(x;x_0,t) = 0$ and therefore $J(x;x_0,t) = 0$. In conclusion

$$I(x;x_0,t) = \inf_{\xi(0)=x_0,\xi(t)=x} \int_0^t \bar{L}\left(\xi(s),\xi'(s)\right) ds.$$

Finally, the claim follows from the continuity of the function $r(y;x_0,t)$ in the variable y, that can be found, e.g., in [70], Section 4, Proposition 3.1 and Corollary 3.4.

□

Out-of-the-money option pricing and asymptotic implied volatility

In this section, we give some applications of Theorem 1.6.2 in dimension 1 to out-of-the-money option pricing. In particular, in Corollary 1.6.5, we state an asymptotic estimate for the behaviour of the price of out-of-the-money European call option with strike price K and short maturity time $T = \varepsilon t$.

Let S_t^ε be the asset price, evolving according to the following stochastic differential system

$$\begin{cases} dS_t^\varepsilon = \varepsilon\xi(S_t^\varepsilon, Y_t^\varepsilon)S_t^\varepsilon dt + \sqrt{2\varepsilon}\zeta(S_t^\varepsilon, Y_t^\varepsilon)S_t^\varepsilon dW_t & S_0^\varepsilon = S_0 \in \mathbb{R}_+ \\ dY_t^\varepsilon = \varepsilon^{1-\alpha}b(Y_t^\varepsilon)dt + \sqrt{2\varepsilon^{1-\alpha}}\tau(Y_t^\varepsilon)dW_t & Y_0^\varepsilon = y_0 \in \mathbb{R}^m, \end{cases} \tag{1.84}$$

where $\alpha > 1$, τ, b are as in (1.6) and $\xi : \mathbb{R}_+ \times \mathbb{R}^m \to \mathbb{R}$, $\zeta : \mathbb{R}_+ \times \mathbb{R}^m \to \mathbf{M}^{1,r}$ are Lipschitz continuous bounded functions, periodic in y. Observe that $S_t^\varepsilon > 0$ almost surely if $S_0 > 0$. We define $X_t^\varepsilon = \log S_t^\varepsilon$. Then $(X_t^\varepsilon, Y_t^\varepsilon)$ satisfies (1.6) with

$$\phi(x,y) = \xi(e^x,y) - \zeta(e^x,y)\zeta^T(e^x,y) \qquad \sigma(x,y) = \zeta(e^x,y).$$

We consider out-of-the-money call option by taking

$$S_0 < K \quad \text{or} \quad x_0 < \log K. \tag{1.85}$$

Following the argument used in [79], we can derive an option price estimates stated in Corollary 1.6.5. Similarly, by considering out-of-the-money put options, one can obtain the same formula for $S_0 > K$.

Corollary 1.6.5. *Suppose that $S_0 < K$. Then, for fixed $t > 0$*

$$\lim_{\varepsilon\to 0^+} \varepsilon \log E\left[(S_t^\varepsilon - K)^+\right] = -\inf_{y > \log K} I(y; x_0, t). \tag{1.86}$$

Now we give an asymptotic estimate of the Black-Scholes implied volatility for out-of-the-money European call option, with strike price K, which we denote by $\sigma_\varepsilon(t, \log K, x_0)$.

We recall that given an observed European call option price for a contract with strike price K and expiration date T, the *implied volatility* σ is defined to be the value of the volatility parameter that must go into the Black-Scholes formula to match the observed price.

By arguments similar to those of the ones used in [79], we get the following asymptotic formula.

Corollary 1.6.6.

$$\lim_{\varepsilon\to 0^+} \sigma_\varepsilon^2(t, \log K, x_0) = \frac{(\log K - x_0)^2}{2\inf_{y>\log K} I(y; x_0, t)t}. \tag{1.87}$$

Note that the infimum in the right-hand side of (1.87), is always positive by assumption (1.85) and by (1.71).

Remark 1.6.7. When $\zeta(s,y) = \zeta(s)$, then thanks to Remark 1.6.3, (1.86) simplifies to

$$\lim_{\varepsilon\to 0^+} \varepsilon \log E\left[(S_t^\varepsilon - K)^+\right] = -I(\log K; x_0, t)$$

and (1.87) reads

$$\lim_{\varepsilon\to 0^+}\sigma_\varepsilon^2(t,\log K,x_0)=\frac{(\log K-x_0)^2}{2I(\log K;x_0,t)t}.$$

Proof. By the definition of implied volatility

$$\begin{aligned} E\left[(S_t^\varepsilon-K)^+\right] &= e^{r\varepsilon t}S_0\Phi\left(\frac{x_0-\log K+r\varepsilon t+\sigma_\varepsilon^2\frac{\varepsilon t}{2}}{\sigma_\varepsilon\sqrt{\varepsilon t}}\right) \\ &- K\Phi\left(\frac{x_0-\log K+r\varepsilon t-\sigma_\varepsilon^2\frac{\varepsilon t}{2}}{\sigma_\varepsilon\sqrt{\varepsilon t}}\right), \end{aligned} \tag{1.88}$$

where Φ is the Gaussian cumulative distribution function. Then the proof follows as in [79], using (1.88) and Corollary 1.6.5. □

Chapter 2

Non compact case

2.1 Introduction

In this chapter we study the asymptotic behaviour as $\varepsilon \to 0$ of fast stochastic volatility systems in the form

$$\begin{cases} dX_t = \varepsilon\phi(X_t,Y_t)dt + \sqrt{2\varepsilon}\sigma(X_t,Y_t)dW_t & X_0 = x \in \mathbb{R}^n, \\ dY_t = \varepsilon^{1-\alpha}b(Y_t)dt + \sqrt{2\varepsilon^{1-\alpha}}\tau(Y_t)dW_t & Y_0 = y \in \mathbb{R}^m, \end{cases} \tag{2.1}$$

where $\varepsilon > 0$, W_t is a standard m-dimensional Brownian motion, the matrix τ is non-degenerate. Note that we do not assume any compactness of the fast variable, which is replaced by some condition implying ergodicity, i.e that the process Y_t has a unique invariant distribution (the long-run distribution) and that in the long term it becomes independent of the initial distribution. In particular, we manage to treat processes mainly of Ornstein-Uhlenbeck type, that is

$$Y_t = (\mu - Y_t)dt + \tau(Y_t)dW_t$$

where $\mu \in \mathbb{R}^m$ is a vector, and τ is bounded and uniformly non-degenerate (see assumption (U), Section 2.1.1). The motivation behind the analysis of such kind of systems relies in the fact that the assumption of periodicity of Chapter 1 seems a bit restrictive for the financial applications we have in mind, in particular it does not appear natural in order to model volatility in financial markets, according to the empirical data and the discussion presented in [86] and the references therein.

Following the line of Chapter 1, we consider a logarithmic functional of the trajectories of (2.1)

$$v^\varepsilon(t,x,y) := \varepsilon \log E\left[e^{h(X_t)/\varepsilon} | (X.,Y.) \text{ satisfy (2.1)}\right],$$

where h is a bounded continuous function and we characterize v^ε as the solution of the Cauchy problem with initial data $v^\varepsilon(0,x,y) = h(x)$ for a fully nonlinear parabolic equation in $n+m$ variables (see Proposition 2.1.5 where we recall this result). Our first aim is to prove that, under suitable assumptions, the functions $v^{\varepsilon}(t,x,y)$ converge to a function $v(t,x)$ characterized as the solution of the Cauchy problem for a first order Hamilton-Jacobi equation in n space dimensions

$$v_t - \bar{H}(x, Dv) = 0 \ \text{ in }]0,T[\times\mathbb{R}^n, \quad v(0,x) = h(x), \tag{2.2}$$

for a suitable effective Hamiltonian $\bar{H}$. We also derive some applications of this result to large deviations, estimation of out-of-the-money option prices near maturity and asymptotic formula for the Black-Scholes implied volatility. Since the proofs of these applications are exactly the same of those shown in Chapter 1, Section 1.6, we omit them.

This is a singular perturbation problem for nonlinear HJB where the fast variable y lives in $\mathbb{R}^m$, which is the main difficulty to deal with. Some of the methods used in Chapter 1 are strongly linked to the assumption of periodicity (that we assumed In Chapter 1) and must be modified in this setting.

The main issues are the resolution of the cell problem, the identification of the limit Hamiltonians and the convergence result. Our methods are based on the use of the approximate δ-cell problem. A key result is the global Lipschitz bound for the solution of the δ-approximate cell problem, uniform in δ, proved in Proposition 2.5.5 (critical case) and Proposition 2.5.14 (supercritical case). The proof is in some part inspired by a method due to Ishii and Lions [112] (see also [67],[28] and the references therein), which essentially allows to take profit of the uniform ellipticity of the equation to control the Hamiltonian terms. However, we remark that usually the Ishii-Lions method allows to achieve bounds which depend on the L^{∞}-norm of the solution (at least if we do not assume any periodicity), whereas our aim is to establish a global estimate in all the space independent of such norm. The fundamental hypothesis which enables us to achieve our result consists in assuming that the fast processes we consider are mainly of Ornstein-Uhlenbeck type. Note also that we deal with both linear Hamiltonians in the gradient (in the supercritical case) and superlinear quadratic Hamiltonians (in the critical case).

In the critical case the proof is carried out in three steps. First we prove an uniform local Lipschitz bound for the solution of the δ-cell problem (see Section 2.5, Lemma 2.5.1). The proof of the local bound is carried out by the Bernstein method relying on the coercivity in the gradient of the cell equation (which, in the critical case, is a uniformly elliptic second order equation with quadratic Hamiltonian in the gradient). Note that, thanks to this local gradient bound, we are able to consider fast processes which coincide with the Ornstein-Uhlenbeck process only outside some ball (see assumption (U), Section 2.1.1). Moreover, it allows us quite general assumptions on the stochastic volatility (see assumption (S), Section 2.1.1). A second step we prove a global Hölder bound not uniform in δ (see Proposition 2.5.2) using the Ishii-Lions method and relying mainly on the uniform ellipticity and assumption (U) . Finally, we achieve the global uniform Lipschitz bound by using both the local bound and Hölder bound already proven and relying deeply on assumptions (U) and (S). We remark that the proof is non standard mainly because we do not use any compactness or periodicity on the coefficients and our result is independent of δ and holds in all the space.

On the contrary, in the supecritical case the cell problem is a uniformly elliptic equation linear in the gradient. Since in this case we are not able to prove an analogous local bound as in Lemma 2.5.1, we streghten assumption (U) and consider processes Y_t which coincide with the Ornstein-Uhlenbeck process in all the space. Note also that in the supercritical case there is no need of assumption (S) on the volatility. For further remarks we refer to Section 2.5, subsection 2.5.2. Mainly beacause assumption (U) holds in all the space (and we do not need (S)), the proofs of the Hölder bound and the global uniform Lipschitz bound in the supercritical case are analogous and even easier than in the critical case.

Let us recall some results in the literature related to gradient bounds for similar kind of equations. Gradient bounds for superlinear-type Hamiltonians can be found in Lions [126] and Barles [20], see also Lions and Souganidis [129] and Barles and Souganidis [32]. Recently, Hölder bounds for nonlinear degenerate parabolic equations were proved in Cardaliaguet and Sylvestre [55]. However, we remark that, in the previous works the bound depends usually on the L^∞-norm of the solution, that is, on δ, when dealing with the δ-cell problem), whereas, on the contrary, our aim is to find a bound which is independent of such parameter. In [32] some results independent of the L^∞ norm of the solutions are established but in periodic environment. We recall also the result of [53] by Capuzzo-Dolcetta, Leoni, Porretta for coercive superlinear Hamiltonians, where a uniform gradient bound is established, but in some Hölder norm and only in bounded domains. We refer also to Barles [21], Cardaliaguet [54]. Recently, uniform Lipschitz bound on the torus for analogous equations as ours (and more general) has been established by Ley and Duc Nguyen in [125].

As already hinted above, the first issue is the identification of the limit Hamiltonian through the resolution of the cell problem which is now defined in all the space. The existence of a limit Hamiltonian and of the corrector is proved by the use of the approximate δ-cell problem. The main result which allows us to conclude the existence is the uniform gradient estimate for the solution of the δ-cell problem. For the uniqueness of the limit Hamiltonian, we proceed differently in the critical and supercritical case. In the critical case, we rely on the ergodicity of the process Y_t and on the results of Ichihara [107], where ergodic type Bellman equations are studied in the case of a nonlinear quadratic term. On the contrary, in the supercritical case, we rely on the results of Bardi, Cesaroni, Manca in [15], where the uniqueness of a limit Hamiltonian is proved (but note that no existence of the true corrector is proved in [15]).

The main result is the convergence of the functions v^ε to the solution of the limit problem. Our techniques are based on the perturbed test function method of [76], [4], with some relevant adaptations to the unbounded setting. In order to deal with the non compactness of the fast variable, we deeply rely on the ergodicity of the fast process, which is encoded in the existence of a *Lyapounov function* (see Section 2.2). A key result used in the convergence is, again, the global gradient bound of the corrector (see Proposition 2.5.6 and Proposition

2.5.15), which we use in the proof of the convergence mainly to deal with the difficulties coming from the nonlinearity in the gradient of our equation.

We recall the paper [79], where Feng, Fouque, and Kumar study analogous problems for system of the form we consider when $\alpha = 2$ and $\alpha = 4$, in the one-dimensional case $n = m = 1$, assuming that Y_t is an Ornstein-Uhlenbeck process and the coefficients in the equation for X_t do not depend on X_t. Their methods are based on the approach to large deviations developed in [80]. We consider $\alpha \geq 2$ and we treat vector-valued processes with ϕ and σ depending on X_t in a rather general way and we study all the range $\alpha \geq 2$; our methods are different, mostly from the theory of viscosity solutions for fully nonlinear PDEs and from the theory of homogenization and singular perturbations for such equations.

2.1.1 The stochastic volatility model

We consider fast mean-reverting processes of the following type

$$\begin{cases} dX_t = \phi(X_t,Y_t)dt + \sqrt{2}\sigma(X_t,Y_t)dW_t, & X_0 = x \in \mathbb{R}^n \\ dY_t = \varepsilon^{-\alpha} b(Y_t)dt + \sqrt{2\varepsilon^{-\alpha}}\tau(Y_t)dW_t, & Y_0 = y \in \mathbb{R}^m, \end{cases} \tag{2.3}$$

where $\varepsilon > 0, \alpha \geq 2, \phi : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n, \sigma : \mathbb{R}^n \times \mathbb{R}^m \to \mathbf{M}^{n,m}$ are bounded functions, Lipschitz continuous in (x,y), $b : \mathbb{R}^m \to \mathbb{R}^m$ is Lipschitz continuous, $\tau : \mathbb{R}^m \to \mathbf{M}^{m,m}$ is bounded, Lipschitz continuous and uniformly non degenerate, i.e. satisfies for some $\theta > 0$

$$\xi^T \tau(y)\tau(y)^T \xi = |\tau^T(y)\xi|^2 > \theta|\xi|^2 \quad \text{for every } y \in \mathbb{R}, \xi \in \mathbb{R}^m. \tag{2.4}$$

This assumptions will hold throughtout the Chapter.

Moreover b, τ satisfy the following condition

(U) There exist $\mu \in \mathbb{R}^m, \tau \in \mathbf{M}^{m,m}$ such that

- If $\alpha = 2$, there exists $R_1 > 0$ such that

  1)
  $$b(y) = \mu - y, \quad \text{if } |y| > R_1;$$

  2)
  $$\tau(y) = \tau \quad \text{if } |y| > R_1;$$

- if $\alpha > 2$
  $$b(y) = \mu - y, \quad \tau(y) = \tau \quad \text{for any } y \in \mathbb{R}^m.$$

We also assume that σ satisfies

(S) for all $x \in \mathbb{R}^n$, there exists $g : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^+$ such that

$$||\sigma(x,y) - \sigma(x,z)||_\infty \le g(y,z)|y-z| \quad \text{for all } y,z \in \mathbb{R}^m.$$

and $\forall \varepsilon > 0$, there exists $R_\varepsilon > 0$ such that $g(y,z) \le \varepsilon$ as $|z|,|y| \ge R_\varepsilon$.

Remark 2.1.1. As already observed, assumption (U) is satisfied in particular by Ornstein-Uhlenbeck type processes. The Ornstein-Uhlenbeck process is a classical example of a Gaussian process that admits a stationary probability distribution. It is a mean-reverting process, namely there is a long-term value μ towards the process "tends to revert". In other words, in dimension $m = 1$, this means that if the current value of the process is less than the (long-term) mean, the drift will be positive; if the current value of the process is greater than the (long-term) mean, the drift will be negative. This gives the process the name "mean-reverting."

Remark 2.1.2. Note that assumption (U) is stronger in the supercritical case $\alpha > 2$. As already note in the introduction, this is due to the fact that in the critical case we can prove a local uniform gradient bound for the solution of the approximate cell problem, whereas in the supercritical case, we are not able to prove such a result. We refer to Section 2.5, subsection 2.5.2 for further remarks.

Remark 2.1.3. Assumption (S) says, roughly speaking, that the Lipschitz constant of $\sigma(x,\cdot)$, considered as a function on $\mathbb{R}^m$ for $x \in \mathbb{R}^n$ fixed, vanishes at infinity. We refer to the following remark for some examples. Note also that (S) plays its role in the proof of the global gradient estimate for the corrector in the critical case, see Proposition 2.5.6. On the contrary, in the supercritical case there is no need of such assumption (see Proposition 2.5.15). We remark that, at least to our point of view, (S) seems not restrictive in the context of financial models, since it influences the behaviour of σ only at infinity, which in general is not "seen" in the financial applications we are interested in.

Remark 2.1.4. Examples of sufficient conditions for (S) are

$$\lim_{|x|\to+\infty} g(x,y) = 0 \quad \text{uniformly in } y,$$

$$\lim_{|y|\to+\infty} g(x,y) = 0 \quad \text{uniformly in } x.$$

For example, the above conditions are satisfied by $\sigma(x,y) = \frac{1}{(1+|y|^2)^\alpha}$, for $\alpha > 0$. Then in this case we have (S) with $g(y,z) = \frac{C}{1+|y|+|z|}$. Without loss of generality we suppose $n = 1$

and $z \geq y \geq 0$. Then

$$\sigma(y) - \sigma(z) = \frac{1}{(1+y^2)^\alpha}\left(1 - \left(1 + \frac{y^2 - z^2}{1+z^2}\right)^\alpha\right).$$

From the inequality $1-(1+x)^\alpha \leq -x$ for $-1 \leq x \leq 0$, we get

$$\sigma(y) - \sigma(z) \leq \frac{1}{(1+y^2)^\alpha}\frac{(z-y)(z+y)}{1+z^2} \leq \frac{2z}{1+z^2}(y-z).$$

Since we assumed $z \geq y \geq 0$, we can find a constant C independent of y,z such that $\frac{2z}{1+z^2} \leq \frac{C}{1+z+y}$, concluding the proof.

In order to study small time behaviour of the system (2.3), we rescale time $t \to \varepsilon t$, for $0 < \varepsilon \ll 1$, so that the typical maturity will be of order ε. Denoting the rescaled process by $X_t^\varepsilon, Y_t^\varepsilon$ we get

$$\begin{cases} dX_t = \varepsilon\phi(X_t,Y_t)dt + \sqrt{2\varepsilon}\sigma(X_t,Y_t)dW_t, & X_0 = x \in \mathbb{R}^n \\ dY_t = \varepsilon^{1-\alpha}b(Y_t)dt + \sqrt{2\varepsilon^{1-\alpha}}\tau(Y_t)dW_t, & Y_0 = y \in \mathbb{R}^m. \end{cases} \tag{2.5}$$

2.1.2 The logarithmic transformation method and the HJB equation

We follow the same approach of Chapter 1. We consider the following functional

$$v^\varepsilon(t,x,y) := \varepsilon \log E\left[e^{h(X_t)/\varepsilon} | (X_\cdot, Y_\cdot) \text{ satisfy (2.5)}\right], \tag{2.6}$$

where $h \in BC(\mathbb{R}^n)$ and (X_s, Y_s) satisfies (2.5). Note that the logarithmic form of this payoff is motivated by the applications to large deviations that we want to give.

Analogously as in Chapter 1 (Proposition 1.1.2), we characterize v^ε as the unique continuous viscosity solution of the following parabolic problem. We refer again to Da Lio and Ley in [69] for a proof.

Proposition 2.1.5. *Let $\alpha \geq 2$ and define*

$$\begin{aligned} H^\varepsilon(x,y,p,q,X,Y,Z) \ &:= \ |\sigma^T p|^2 + b\cdot q + tr(\tau\tau^T Y) + \varepsilon\left(tr(\sigma\sigma^T X) + \phi\cdot p\right) \\ &+ \ 2\varepsilon^{\frac{\alpha}{2}-1}(\tau\sigma^T p)\cdot q + 2\varepsilon^{\frac{1}{2}} tr(\sigma\tau^T Z) + \varepsilon^{\alpha-2}|\tau^T q|^2. \end{aligned}$$

Then v^ε is the unique bounded continuous viscosity solution of the Cauchy problem

$$\begin{cases} \partial_t v^\varepsilon - H^\varepsilon\left(x,y,D_x v^\varepsilon, \frac{D_y v^\varepsilon}{\varepsilon^{\alpha-1}}, D^2_{xx}v^\varepsilon, \frac{D^2_{yy}v^\varepsilon}{\varepsilon^{\alpha-1}}, \frac{D^2_{xy}v^\varepsilon}{\varepsilon^{\frac{\alpha-1}{2}}}\right) = 0 & in\ [0,T]\times\mathbb{R}^n\times\mathbb{R}^m, \\ v^\varepsilon(0,x,y) = h(x) & in\ \mathbb{R}^n\times\mathbb{R}^m. \end{cases} \tag{2.7}$$

Remark 2.1.6. We observe that we treat the supercritical ($\alpha > 2$) and critical ($\alpha = 2$) case and we do not deal with the subcritical case ($\alpha < 2$). Indeed, for $\alpha < 2$, the cell problem is finding, for each $(\bar{x}, \bar{p})$ fixed, a unique constant $\bar{H}(\bar{x}, \bar{p})$ such that the following equation has a viscosity solution w:

$$\bar{H}(\bar{x}, \bar{p}) - 2(\tau(y)\sigma(\bar{x}, y)^T \bar{p}) \cdot D_y w(y) - |\tau(y)^T D_y w(y)|^2 - |\sigma(\bar{x}, y)^T \bar{p}|^2 = 0. \tag{2.8}$$

Note that in this case the cell problem is not solvable in general. This is essentially due to the fact that the ergodicity of the fast process plays no role in the cell problem (2.8), since the cost ($|\sigma^T \bar{p}|^2$) and the drift $2\tau\sigma^T \bar{p}$ are both bounded and the drift b has disappeared. On the contrary, in the case $\alpha \geq 2$, this role is played by the term $-b \cdot Dw$ where b satisfies assumption (U). We refer to the following subsection where we use explicitly the condition on b of (U), in the case $\alpha \geq 2$, in order to prove the existence of a Liapounov function. We refer also to [107], where analogous conditions to (U), (even more general, see for example (2.9) and (2.14) in the following subsection) are assumed to ensure the ergodicity.

Remark 2.1.7. As a consequence of the convergence result, we prove a large deviation principle for the process X_t^ε whose statement and proof we omit in this chapter, since they are analogous to those of Theorem 1.6.2 of Chapter 1, Section 1.6. The same remark holds for the asymptotic estimate for out-of-the-money option prices near maturity and asymptotic formula for the implied volatility proved in Corollary 1.6.5 and Corollary 1.6.6 of Chapter 1, Section 1.6.

2.2 A Liapounov-like condition

We prove that, under the assumption (U), there exists for any $(\bar{x}, \bar{p}) \in \mathbb{R}^n \times \mathbb{R}^n$ a Liapounov-like function for the following operator

$$\mathscr{G}_{\bar{x},\bar{p}}(y, q, Y) = -(b(y) + 2\tau(y)\sigma^T(\bar{x}, y)\bar{p}) \cdot q - |\tau^T(y)q|^2 - \mathrm{tr}(\tau\tau^T(y)Y),$$

i.e. we prove that there exists a continuous function $\chi_{\bar{x},\bar{p}} := \chi$, such that $\chi(y) \to +\infty$ as $|y| \to +\infty$ and if $\mathscr{G}[\chi] := \mathscr{G}_{\bar{x},\bar{p}}(y, D\chi(y), D^2\chi(y))$ then

$$\mathscr{G}[\chi] \to +\infty \text{ as } |y| \to +\infty \text{ in the viscosity sense.} \tag{2.9}$$

The existence of a Liapounov function is reminiscent of other similar conditions about ergodicity of diffusion processes in the whole space; see, for example [105],[127], [36], [45], [132].

Remark 2.2.1. We observe that

$$\mathscr{G}_{\bar{x},\bar{p}}(y,q,Y) = -\mathscr{L}_{\bar{x},\bar{p}}(y,q,Y) - |\tau^T(y)q|^2 \tag{2.10}$$

where, for any $(\bar{x},\bar{p}) \in \mathbb{R}^n \times \mathbb{R}^n$, $\mathscr{L}_{\bar{x},\bar{p}}$ is the linear operator

$$\mathscr{L}_{\bar{x},\bar{p}}(y,q,Y) = (b(y) + 2\tau(y)\sigma^T(\bar{x},y)\bar{p}) \cdot q + \mathrm{tr}(\tau\tau^T(y)Y),$$

which is the infinitesimal generator of the stochastic process

$$dY_t = (b(Y_t) + 2\tau(Y_t)\sigma^T(\bar{x},Y_t)\bar{p})dt + \tau(Y_t)dW_t.$$

Note that we consider the additional term $-|\tau^T q|^2$ in (2.10) and this is due to the logarithmic form of the value function v_ε defined in (2.6), which is in turn motivated by the applications to large deviations we are interested in.

Now we prove the existence of a Lyapounov function for the operator $\mathscr{G}$.

Proof. Note that a key role in the following proof is played by the behavior of the drift b at infinity, which is encoded by the condition on b of assumption (U).

We take

$$\chi = a|y|^2, \tag{2.11}$$

and by (U), and the boundedness of τ, we have for $|y| \geq R_1$

$$-b(y) \cdot D\chi(y) - |\tau^T(y)D\chi(y)|^2 \geq 2a|y|^2 - 4a^2T|y|^2 - 2a|\mu||y|, \tag{2.12}$$

where $T > 0$ depends on $||\tau||_\infty$. Then by taking

$$a < \frac{1}{2T}, \tag{2.13}$$

the other terms in $\mathscr{G}$ being negligible because of the boundedness of τ and σ, we get (2.9).

□

Remark 2.2.2. We remark that the previous proof holds under a weaker condition on b, which is implied by (U), namely

$$b(y) \cdot y \leq -B|y|^2 \quad \text{for } |y| \geq R_2 \text{ for some } R_2 > 0, B > 0. \tag{2.14}$$

We observe that condition (2.14) reminds classical conditions for ergodicity, see for example [15]. In particular we recall the so-called *recurrence* condition used by Pardoux and

Veretennikov [134], [135], [136] namely

$$b(y) \cdot y \to -\infty \quad \text{as } |y| \to +\infty. \tag{2.15}$$

Note that (2.14) is stronger than (2.15). The main reason is that in our context we need to have some additional information on the rate of decay of $b \cdot y$, in particular we need it to be at least quadratic in order to compete with the quadratic growth (in the gradient term) of $\mathscr{G}$ (see also Remark 2.2.1). We note that we assume condition (U), which is stronger than (2.14), for reasons linked to the convergence of the functional v^{ε} and to the proof of the global gradient bound for the corrector of Proposition 2.5.6 and Proposition 2.5.15.

2.3 The critical case: $\alpha = 2$

2.3.1 Key preliminary results

For any $(\bar{x}, \bar{p}) \in \mathbb{R}^n \times \mathbb{R}^n$, the cell problem is finding the unique constant $\lambda \in \mathbb{R}$ such that the following equation

$$\lambda - \operatorname{tr}(\tau\tau^T(y)D^2w(y)) - |\tau^T(y)Dw|^2 - (b(y) + 2(\tau(y)\sigma^T(\bar{x}, y)\bar{p}) \cdot Dw(y) - |\sigma^T \bar{p}|^2 = 0. \tag{2.16}$$

has a viscosity solution w. This kind of ergodic problems have been studied by Ichihara [107] and Ichihara and Sheu [108]. We refer in particular to Theorem 2.4 of [107], which we recall in the following proposition.

Denote

$$\Phi = \{w \in C^2(\mathbb{R}^m) : \text{ there exists } C < 0 \text{ such that } w(y) \le C(1 + |y|)\}. \tag{2.17}$$

Proposition 2.3.1. *Let assumption 1) of (U) holds. There exists a constant $\lambda^* \in \mathbb{R}$ such that* (2.16) *admits a classical solution $w \in C^2(\mathbb{R}^m)$ if and only if $\lambda \le \lambda^*$. Moreover, if (λ, w) is a solution of* (2.16) *and $w \in \Phi$, then $\lambda = \lambda^*$.*

Remark 2.3.2. We remark that Theorem 2.4 is proved for Hamiltonians which are convex in the gradient variable, whereas in our case the Hamiltonian is concave. The two cases are equivalent, since if we have a solution w of (2.16), then $-w$ is a solution of

$$-\lambda - \operatorname{tr}(\tau\tau^T(y)D^2w(y)) + H(y, Dw(y)) = 0, \tag{2.18}$$

where

$$H(y,q) = -b(y) \cdot q + |\tau(y)^T q|^2 - 2\tau(y)\sigma(\bar{x}, y)^T \bar{p} \cdot q + |\sigma(\bar{x}, y)\bar{p}|^2. \tag{2.19}$$

which is now convex in the gradient and satisfies the assumptions of [107].

2.3.2 The cell problem and the effective Hamiltonian

For $\delta > 0$, we consider the approximate cell problem

$$\delta w_\delta + F(\bar{x}, y, \bar{p}, Dw_\delta, D^2 w_\delta) - |\sigma(\bar{x}, y)\bar{p}|^2 = 0, \tag{2.20}$$

where

$$F(\bar{x}, y, \bar{p}, q, Y) := -\mathrm{tr}(\tau\tau^T(y)Y) - |\tau^T(y)q|^2 - b(y)\cdot q - 2(\tau(y)\sigma^T(\bar{x}, y)\bar{p})\cdot q. \tag{2.21}$$

Under our standing assumptions we have the following results.

Proposition 2.3.3. *Let assumption 1) of (U) holds. For any $(\bar{x}, \bar{p})$ fixed, there exists a unique solution $w_\delta \in C^2(\mathbb{R}^m)$ of* (2.20) *satisfying*

$$-\frac{1}{\delta}\inf_{y\in\mathbb{R}^m} |\sigma(\bar{x}, y)^T \bar{p}|^2 \le w_\delta(y) \le \frac{1}{\delta}\sup_{y\in\mathbb{R}^m} |\sigma(\bar{x}, y)^T \bar{p}|^2, \tag{2.22}$$

such that

$$\lim_{\delta\to 0} \delta w_\delta(y) = \ const \ := \bar{H}(\bar{x}, \bar{p}) \text{ locally uniformly.}$$

Moreover $\bar{H}(\bar{x}, \bar{p})$ is the unique constant such that (2.16) *has a solution $w \in C^2(\mathbb{R}^m)$ satisfying*

$$|w(y)| \le \bar{C}(1 + \log(\sqrt{|y|^2 + 1})) \quad \text{for all } y \in \mathbb{R}^m. \tag{2.23}$$

Finally w is the unique (up to and additive constant) solution to (2.16) *for $\lambda = \bar{H}(\bar{x}, \bar{p})$.*

Remark 2.3.4. The growth estimate (2.23) implies that w solution of (2.16) belongs to the class Φ defined in (2.17), allowing us to apply Proposition 2.3.1 and deriving the uniqueness of $\bar{H}$. Note that (2.23) is stronger than the growth required in Φ, in particular it would be enough to prove (2.23) with a linear function of y in the right-hand side.

Remark 2.3.5. We remark that in the following proof we will use a local gradient bound uniform in δ for the solution of the approximate cell problem. We state and prove the result in Section 2.5, see Lemma 2.5.1.

Now we prove Proposition 2.3.3.

Proof of Proposition 2.3.3. We split the proof into two steps. In step 1 we prove the existence of a couple $(w, \lambda) \in C(\mathbb{R}^m) \times \mathbb{R}$ solution to (2.16); in step 2 we prove that $w \in C^2(\mathbb{R}^m)$, (2.23) and the uniqueness of such λ. Note that the uniqueness up to an additive constant of w follows from Theorem 2.2 of [107].

Step. *1-Existence* We use the methods of [9] based on the small discount approximation (2.20). Note that the PDE (2.20) has bounded forcing term $|\sigma^T(\bar{x},y)\bar{p}|^2$ since σ is bounded. The existence and uniqueness of a viscosity solution with the δ dependent bound (2.22) follows from the Perron-Ishii method and the comparison principle in [69]. Moreover $w_\delta \in C^2(\mathbb{R}^m)$, thanks to the Lipschitz uniform estimate of Lemma 2.5.1 and by elliptic regularity theory of convex uniformly elliptic equations, see [153] and [141].

Now we prove that $\delta w_\delta(y)$ converges along a subsequence of $\delta \to 0$ to the constant $\bar{H}(\bar{x},\bar{p})$ and $w_\delta(y) - w_\delta(0)$ converges to the corrector w. The hard part is proving equicontinuity estimates for δw_δ. We proceed by a diagonal argument. By the local Lipschitz estimates of Lemma 2.5.1, we have

$$|w_\delta(y) - w_\delta(z)| \le C_1(1+|\bar{p}|)|y-z| \quad y,z \in \bar{B}_1, \tag{2.24}$$

that is, δw_δ is equicontinuous in $\bar{B}_1$. The equiboundedness follows from the comparison principle with constant sub and super solutions, namely $\min_{y\in\mathbb{R}^m} |\sigma(y,\bar{x})^T\bar{p}|^2$ and $\max_{y\in\mathbb{R}^m} |\sigma(y,\bar{x})^T\bar{p}|^2$. Then by Ascoli-Arzela theorem, there exists a subsequence $\delta_n^1 w_{\delta_n^1}$ of δw_δ, converging uniformly in $\bar{B}_1$ to a constant λ^1, since by (2.24) we have

$$|\delta w_\delta(y) - \delta w_\delta(z)| \le \delta C_1(1+|\bar{p}|)|y-z| \quad y,z \in \bar{B}_1$$

and then

$$\delta w_\delta(y) - \delta w_\delta(z) \to 0 \quad \forall y,z \in \bar{B}_1 \text{ as } \delta \to 0.$$

By the same argument, $\delta_n^1 w_{\delta_n^1}$ is equibounded and equicontinous in $\bar{B}_2$. Then, there exists a subsequence $\delta_n^2 w_{\delta_n^2}$ of $\delta_n^1 w_{\delta_n^1}$, converging uniformly in $\bar{B}_2$ to a constant λ^2, such that

$$\lambda^1 = \lambda^2 =: \lambda.$$

Similarly, we construct for all $k \in \mathbb{N}$, a sequence $\{\delta_n^k w_{\delta_n^k}\}_n$ converging as $n \to \infty$ uniformly in $\bar{B}_k$ to a constant $\lambda^k = \lambda$. Note that the subsequence $\{\delta_n^n w_{\delta_n^n}\}_n$ converges locally uniformly to λ. In fact for any $k \in \mathbb{N}$ we have that $\{\delta_n^n w_{\delta_n^n}\}_n$ is a subsequence of $\{\delta_n^k w_{\delta_n^k}\}_n$ for all $n \ge k$, from which we deduce that $\{\delta_n^n w_{\delta_n^n}\}_n$ converges uniformly in $\bar{B}_k$ for all $k \in \mathbb{N}$.

Now define $v_\delta := w_\delta(y) - w_\delta(0)$. Notice that, for all k, v_δ is equibounded in $\bar{B}_k$, since, by Lemma 2.5.1, we have

$$|v_\delta(y)| = |w_\delta(y) - w_\delta(0)| \le C_k(1+|\bar{p}|)|y|, \quad y \in \bar{B}_k$$

and, again by Lemma 2.5.1, v_δ is equicontinuous in B_k since

$$|v_\delta(y) - v_\delta(z)| = |w_\delta(y) - w_\delta(z)| \le C_k(1+|\bar{p}|)|y-z|, \quad y,z \in \bar{B}_k.$$

By an analogous diagonal argument, we find sequences $\{v_{\delta_n^k}\}_n$ such that $v_{\delta_n^{k+1}}$ is a subsequence of $v_{\delta_n^k}$, and converges uniformly in $\bar{B}_k$ to a function v^k. Moreover for all $k \in \mathbb{N}$, we have

$$v^{k+1}(y) = v^k(y) \quad y \in \bar{B}_k.$$

Then, if we define $w : \mathbb{R}^m \to \mathbb{R}$ such that

$$w(y) = v^k(y) \quad y \in \bar{B}_k, \tag{2.25}$$

we conclude that

$$\{v_{\delta_n^n}\}_n \to w \quad \text{locally uniformly.} \tag{2.26}$$

Now we prove that (λ, w) satisfy (2.16). From (2.20) we get

$$\delta v_\delta + \delta w_\delta(0) + F(\bar{x}, y, \bar{p}, D_y v_\delta, D^2_{yy} v_\delta) - |\sigma^T(\bar{x}, y)\bar{p}|^2 = 0, \quad \text{in } \mathbb{R}^m. \tag{2.27}$$

Since v_δ is locally equibounded, $\delta v_\delta \to 0$ locally uniformly and the claim follows recalling that $\delta w_\delta \to \lambda$ and using the stability property of viscosity solutions.

Finally the corrector inherits the property (2.40) of Lemma 2.5.1 and satisfies for all $k \in \mathbb{N}$

$$\max_{y \in \bar{B}(0,k)} |D_y w(y; \bar{x}, \bar{p})| \le C_k(1 + |\bar{p}|), \tag{2.28}$$

for some $C_k > 0$ and for all $\bar{x}, \bar{p} \in \mathbb{R}^n$.

Step. *2-Uniqueness of* λ The uniqueness is given by Proposition 2.3.1, once proved that $w \in \Phi$. The C^2 regularity follows from the uniform Lipschitz estimate (2.28) and the regularity theory of convex uniformly elliptic equations, see [153] and [141].

Note that, in order to prove that $w \in \Phi$, we prove the (stronger) growth condition (2.23).

We prove the claim for the upper bound, since the proof of the lower bound is analogous.

We take the approximate problem (2.20) and we prove that the function $g = C\log(\sqrt{|y|^2+1})$, for some positive constant C large enough, is a supersolution of (2.20), that is, we prove

$$\delta g(y) - (b(y) + 2\tau(y)\sigma^T(\bar{x}, y)\bar{p}) \cdot Dg - |\tau^T(y) Dg(y)|^2 - \mathrm{tr}(\tau\tau^T(y) D^2 g) - |\sigma(\bar{x}, y)\bar{p}|^2 \ge 0. \tag{2.29}$$

Take $|y| \ge R_1$ where R_1 is defined in (U). By (U) and the boundedness of σ, we have

$$\delta g(y) - (b(y) + 2\tau(y)\sigma^T(\bar{x}, y)\bar{p}) \cdot Dg - |\tau^T(y) Dg(y)|^2 - \mathrm{tr}(\tau\tau^T(y) D^2 g) - |\sigma(\bar{x}, y)\bar{p}|^2 \ge$$
$$2C\frac{|y|^2}{|y|^2+1} - \frac{KC(1+|\bar{p}|)|y|}{|y|^2+1} - KC^2\frac{|y|^2}{(|y|^2+1)^2} - |\sigma(\bar{x}, y)\bar{p}|^2, \tag{2.30}$$

where $K > 0$ depends on τ, μ defined in (U) and on $||\sigma(\bar{x}, \cdot)||_\infty$. Then, in order to prove that g is a supersolution of (2.29), we prove that the second term in (2.30) is non negative. We factorise $\frac{|y|^2}{|y|^2+1}$ and we prove that

$$C - \frac{KC(1+|\bar{p}|)}{|y|} - \frac{KC^2}{|y|^2+1} - \sup_y |\sigma^T \bar{p}|^2 \frac{|y|^2+1}{|y|^2} \geq 0. \tag{2.31}$$

Note that when y goes to infinity in (2.31) the leading order term is $C - \sup_y |\sigma^T \bar{p}|^2$. Then the claim follows by taking for example $C = 2 + \frac{3}{2} \sup_y |\sigma^T \bar{p}|^2$ and $y \in \mathbb{R}^m \setminus \bar{B}_{\bar{R}}$ for some $\bar{R} > R_1$ such that

$$\frac{KC(1+|\bar{p}|)}{|y|} + \frac{KC^2}{|y|^2+1} \leq 2, \quad \frac{|y|^2+1}{|y|^2} \leq \frac{3}{2}.$$

Up to now we proved that the function $C\log(\sqrt{|y|^2+1})$ is a supersolution of (2.20) in $\mathbb{R}^m \setminus B_{\bar{R}}$. If $\max_{\bar{B}_{\bar{R}}} w_\delta \leq 0$ then

$$w_\delta(y) \leq \max_{\bar{B}_{\bar{R}}} w_\delta \leq C\log(\sqrt{|y|^2+1}) \quad y \in \partial B_{\bar{R}},$$

and then by the comparison principle we have

$$w_\delta(y) \leq C\log(\sqrt{|y|^2+1}) \quad y \in \mathbb{R}^m.$$

Now suppose that $\max_{\bar{B}_{\bar{R}}} w_\delta \geq 0$ and notice that in this case $C\log(\sqrt{|y|^2+1}) + \max_{\bar{B}_{\bar{R}}} w_\delta$ is still a supersolution of (2.20) in $\mathbb{R}^m \setminus B_{\bar{R}}$. Then, again by the comparison principle, we get

$$w_\delta(y) \leq C\log(\sqrt{|y|^2+1}) + \max_{\bar{B}_{\bar{R}}} w_\delta \quad y \in \mathbb{R}^m. \tag{2.32}$$

Since w_δ satisfies (2.32)

$$v_\delta(y) = w_\delta(y) - w_\delta(0) \leq C\log(\sqrt{|y|^2+1}) + \max_{\bar{B}_{\bar{R}}} w_\delta(y) - w_\delta(0) \quad y \in \mathbb{R}^m.$$

We estimate the term $\max_{\bar{B}_{\bar{R}}} w_\delta(y) - w_\delta(0)$ by Lemma 2.5.1 and we get

$$v_\delta(y) \leq C\log(\sqrt{|y|^2+1}) + C_{\bar{R}}$$

and thanks to (2.26) we conclude (2.23) by taking $\bar{C} = \max\{C, C_{\bar{R}}\}$.

□

We recall some properties satisfied by $\bar{H}$. Note that the following proposition can be proved as Proposition 1.2.3 of Chapter 1.

Proposition 2.3.6. *Let assumption 1) of (U) holds.*

(a) $\bar{H}$ *is continuous on* $\mathbb{R}^n \times \mathbb{R}^n$;

(b) *the function* $p \to \bar{H}(x,p)$ *is convex;*

(c)
$$\inf_{y\in\mathbb{R}^m} |\sigma^T(\bar{x},y)\bar{p}|^2 \le \bar{H}(\bar{x},\bar{p}) \le \sup_{y\in\mathbb{R}^m} |\sigma^T(\bar{x},y)\bar{p}|^2; \tag{2.33}$$

(d) *For all* $0<\mu<1$ *and* $x,z,q,p \in \mathbb{R}^n$*, it holds*

$$\mu\bar{H}\left(x,\frac{p}{\mu}\right) - \bar{H}(z,q) \ge \frac{1}{\mu-1}\sup_{y\in\mathbb{R}^m} |\sigma^T(x,y)p - \sigma^T(z,y)q|^2. \tag{2.34}$$

Finally we observe that equations like (2.16) have been studied in a non compact setting by Khaise and Sheu in [116]. They prove the existence of a constant $\bar{H}$ such that there is a unique smooth solution w of (2.20) with prescribed growth. Moreover they provide a representation formula for $\bar{H}$ as the convex conjugate of a suitable operator over a space of measures.

2.4 The supercritical case: $\alpha > 2$

The cell problem is finding, for any $(\bar{x},\bar{p}) \in \mathbb{R}^n \times \mathbb{R}^n$ fixed, a unique constant $\lambda \in \mathbb{R}$ such that the following uniformly elliptic linear equation has a viscosity solution w

$$\lambda - \mathrm{tr}(\tau\tau^T(y)D^2w(y)) - b(y)\cdot Dw(y) - |\sigma(\bar{x},y)^T\bar{p}|^2 = 0. \tag{2.35}$$

This kind of cell problems has been studied in [15], see in particular Proposition 4.2 and Theorem 4.3.

Proposition 2.4.1. *Let assumption (U) holds. For any* $(\bar{x},\bar{p}) \in \mathbb{R}^n \times \mathbb{R}^n$*, there exists a unique invariant probability measure* μ *for the process*

$$dY_t = b(Y_t)dt + \sqrt{2}\tau(Y_t)dW_t. \tag{2.36}$$

Remark 2.4.2. For the details we refer to [15], Proposition 4.2. We just observe that the proof relies strongly on the existence of a Liapounov function as proved in the paragraph 2.2 for the infinitesimal generator of the process (2.36), that is, the operator $\mathscr{L}(y,q,Y) = \mathrm{tr}(\tau\tau^T(y)Y) - b(y)\cdot q$.

Consider the δ-*cell problem* for fixed $(\bar{x},\bar{p},\bar{X})$

$$\delta w_\delta(y) - |\sigma(\bar{x},y)^T\bar{p}|^2 - b(y)\cdot D_y w_\delta(y) - \mathrm{tr}(\tau(y)\tau(y)^T D^2_{yy} w_\delta(y)) = 0 \text{ in } \mathbb{R}^m. \tag{2.37}$$

We have the following proposition, which is analogous to Proposition 2.3.3. We omit the proof since it is the same of Proposition 2.3.3 and even simpler. Note only that in the place of Lemma 2.5.1 we use Proposition 2.5.14 which we prove in Section 2.5.

Proposition 2.4.3. *Let assumption (U) holds. For any fixed* $(\bar{x}, \bar{p})$ *there exists a unique solution* $w_\delta \in C^2(\mathbb{R}^m)$ *satisfying*

$$-\frac{1}{\delta}\inf_{y\in\mathbb{R}^m} |\sigma(\bar{x},y)^T\bar{p}|^2 \leq w_\delta(y) \leq \frac{1}{\delta}\sup_{y\in\mathbb{R}^m} |\sigma(\bar{x},y)^T\bar{p}|^2 \tag{2.38}$$

such that

$$\bar{H}(\bar{x},\bar{p}) = \lim_{\delta\to 0} \delta w_\delta(y), \text{ locally uniformly .}$$

Moreover $\bar{H}(\bar{x},\bar{p})$ *is the unique constant such that* (2.35) *has a viscosity solution* $w \in C^2(\mathbb{R}^m)$ *satisfying* (2.23)*. Finally w is unique up to and additive constant.*

We observe that $\bar{H}$ satisfies the properties $(a),(b),(c),(d)$ of Proposition 2.3.6, which can be proved with similar arguments.

We have the following representation formula for $\bar{H}$. For the proof we refer to [15], Theorem 4.3.

Proposition 2.4.4. *Let assumption (U) holds. For any* $(\bar{x},\bar{p})$ *fixed, let* $\bar{H}(\bar{x},\bar{p})$ *be defined as in Proposition 2.4.3. Then*

$$\bar{H}(\bar{x},\bar{p}) := \int_{\mathbb{T}^m} |\sigma(\bar{x},y)^T\bar{p}|^2\, d\mu(y), \tag{2.39}$$

where μ *is the unique invariant probability measure of the process* (2.36).

2.5 Gradient bounds

The aim of this section is to prove global uniform Lipschitz bounds for the solution of the cell problem both in the critical case and in the supercritical case. This is a key property on which we strongly rely in the proof of the convergence result in Section 2.7. First we analyse the critical case, where we carry out all the computations. Since in the supercritical case the proof is similar and even easier, we give some details in subsection 2.5.2 showing the main differences, but without repeating all the computations.

2.5.1 Critical case

In the critical case, the strategy of the proof consists, roughly speaking, in three steps. The first step consists in proving a Lipschitz bound on compact sets, uniform in δ, for the solution of the δ-cell problem (2.20), which we state in Lemma 2.5.1. The proof of Lemma 2.5.1 is based on the Berstein method and mainly relies on the presence of the coercive term in the gradient $|\tau^T Dw_\delta|^2$ in the equation.

The second step consists in proving an Hölder bound not uniform in δ (see Proposition 2.5.2). The method is essentially based on the Ishii-Lions method and relies mainly on the uniform ellipticity of the equation.

Finally, as a third step, we prove the global uniform gradient bound in Proposition 2.5.5. Note that Lemma 2.5.1 allows us in the critical case to weaken the hypothesis on the process Y_t and consider processes satisfying (U) and (S). The proof of Proposition 2.5.5 relies mainly on the Hölder bound of Proposition 2.5.2 and on the use of assumptions (U) and (S).

Local uniform Lipschitz bound

The first result is the local uniform Lipschitz bound for the solution of the δ-cell problem in the critical case. Note that in Chapter 1, Lemma 1.2.2, we proved the result by the Bernstein method under the assumption of periodicity; the extension to a local bound follows by cut-off functions arguments, following the derivation of similar estimate in [78]. We refer also to [116], Lemma 2.4 for an analogous result. We only note that a key role in the proof of the lemma is played by the quadratic term in the gradient $|\tau^T Dw_\delta|^2$ present in the cell problem in the critical case.

Lemma 2.5.1. *Let $\delta > 0$ and $w_\delta(\cdot;\bar{x},\bar{p}) \in C^2(\mathbb{R}^m)$ be a solution* (2.20)*. Then for all $k \in \mathbb{N}$, there exists $C_k > 0$ such that for all $\bar{x},\bar{p} \in \mathbb{R}^n$ it holds*

$$\max_{y\in\bar{B}(0,k)} |D_y w_\delta(y;\bar{x},\bar{p})| \leq C_k(1+|\bar{p}|). \tag{2.40}$$

Global Hölder bounds

We prove the following Hölder bound in Proposition 2.5.2.

The proof is based on the Ishii-Lions method which allows us to take profit of the uniform ellipticity. As usual in the Ishii-Lions method, the estimate that we prove in (2.41) is not uniform in δ. This is the main difference between Proposition 2.5.2 and Proposition 2.5.5 and, mainly for this reason, the proof of Proposition 2.5.2 is more standard.

We remark that in the following proof we do not need assumption (S), which, on the contrary, is fundamental in the proof of Proposition 2.5.5.

Proposition 2.5.2. *Let assumption (U) holds. Let $w_\delta(\cdot;\bar{x},\bar{p}) \in C^2(\mathbb{R}^m)$ be the solution of* (2.20) *(satisfying the bound* (2.22)*). Then there exists $C_\delta > 0$ and $\alpha \in (0,1)$ such that*

$$|w_\delta(x;\bar{x},\bar{p}) - w_\delta(y;\bar{x},\bar{p})| \leq C_\delta|x-y|^\alpha \quad \textit{for all } x,y \in \mathbb{R}^m, \tag{2.41}$$

where C_δ depends on $\delta, \alpha, ||\tau||_\infty, ||\sigma(\bar{x}, \cdot)||_\infty, \bar{p}$, the Lipschitz constants of τ, b, σ and θ of (2.4).

Proof. Note that throughout the following proof we denote either by (a,b) or $a \cdot b$ the scalar product for any $a, b \in \mathbb{R}^m$.

For convenience of notation in the following we drop the dependence on $\bar{x}, \bar{p}$ by denoting the solution of (2.20) by w_δ.

Let $\delta > 0$ and $\alpha \in (0,1)$ be fixed and consider the function

$$w_\delta(x) - w_\delta(y) - C_\delta |x-y|^\alpha, \tag{2.42}$$

for some constant $C_\delta > 0$ large enough. Note that C_δ will be choosen suitably at the end of the proof and will depend on $\delta, \alpha, ||\tau||_\infty, ||\sigma(\bar{x}, \cdot)||_\infty, \bar{p}$, the Lipschitz constants of τ, b, σ and θ of (2.4). For clearness of exposition, we keep track only of the dependence on δ.

We suppose that

$$\sup\{w_\delta(x) - w_\delta(y) - C_\delta |x-y|^\alpha\} = M > 0.$$

Let $R > 0$ and consider the function

$$\Phi(x,y) = w_\delta(x) - w_\delta(y) - C_\delta |x-y|^\alpha - \psi_R(x) - \psi_R(y), \tag{2.43}$$

where

$$\psi_R(z) = \psi\left(\frac{\sqrt{|z|^2+1}}{R}\right) \tag{2.44}$$

and $\psi \in C^2([0,+\infty))$ satisfies

$$\begin{cases} \psi(s) = 2||w_\delta||_\infty + 1 & \text{if } s \geq 1 \\ \psi(0) = 0, \ \psi \geq 0, \ \psi' \geq 0, \end{cases} \tag{2.45}$$

where we note that $||w_\delta||_\infty$ depends on δ as in (2.22). We claim that

$$M_R = \sup \Phi(x,y) \to M \text{ as } R \to +\infty. \tag{2.46}$$

In fact

$$M_R \leq M \qquad \text{for any } R > 0.$$

On the other hand

$$M_R \geq w_\delta(x) - w_\delta(y) - C_\delta |x-y|^\alpha - \psi_R(x) - \psi_R(y) \ \text{ for all } x, y \in \mathbb{R}^m, R > 0,$$

then

$$\lim_{R \to +\infty} M_R \geq w_\delta(x) - w_\delta(y) - C_\delta |x-y|^\alpha \ \text{ for all } x, y \in \mathbb{R}^m$$

and we conclude

$$\lim_{R\to+\infty} M_R \geq \sup\{w_\delta(x) - w_\delta(y) - C_\delta|x-y|^\alpha\} = M.$$

Then we can suppose for R large enough

$$M_R \geq \frac{M}{2} > 0. \tag{2.47}$$

We observe that if $\sqrt{|x|^2+1} \geq R$

$$\Phi(x,y) \leq -1 < 0$$

and the same holds when $\sqrt{|y|^2+1} \geq R$. Then, there exists (x_R, y_R) point of maximum of Φ such that

$$M_R = w_\delta(x_R) - w_\delta(y_R) - C_\delta|x_R - y_R|^\alpha - \psi_R(x_R) - \psi_R(y_R). \tag{2.48}$$

Note that (x_R, y_R) depends also on δ and that we omit the dependence. Note also that

$$|x_R - y_R| > 0, \tag{2.49}$$

otherwise by (2.48) we have

$$M_R = -\psi_R(x_R) - \psi_R(y_R)$$

and we get a contradiction by (2.47) and the definition of ψ_R.

By (2.46), (2.47) and the definition of ψ_R, we also have

$$C_\delta|x_R - y_R|^\alpha \leq 2||w_\delta||_\infty := A_\delta.$$

Then

$$|x_R - y_R| \leq \left(\frac{A_\delta}{C_\delta}\right)^{\frac{1}{\alpha}}. \tag{2.50}$$

From now on we omit the dependence on R and we write

$$(x_R, y_R) = (x, y).$$

The main result is the following lemma.

Lemma 2.5.3. *Under the above notations and assumption (U), there exist positive constants K, K_1, K_2, K_3, K_4 such that*

$$0 \le KC_\delta \alpha(\alpha-1)|x-y|^{\alpha-2} + KC_\delta \alpha |x-y|^{\alpha+1} + K_1 C_\delta \alpha |x-y|^{\alpha} + K_2 \alpha C_\delta^2 |x-y|^{2\alpha-1} + K_2 o_R(1) C_\delta \alpha |x-y|^{\alpha-1} + K_3 \alpha C_\delta |x-y|^{\alpha-1} + K_4 |x-y| + o_R(1).$$

where by $o_R(1)$ we mean that $o_R(1) \to 0$ as $R \to +\infty$. Moreover K, K_1, K_2, K_3, K_4 depends only on $\bar{p}, ||\sigma||_\infty, ||\tau||_\infty$, the Lipschitz constants of τ, b, σ and θ of (2.4).

Proof. Let

$$r_x = D_x \psi_R = 2R^{-1} \psi' \left(\frac{\sqrt{|x|^2+1}}{R} \right) x \sqrt{|x|^2+1}^{-1} \tag{2.51}$$

and

$$r_y = D_y \psi_R = 2R^{-1} \psi' \left(\frac{\sqrt{|y|^2+1}}{R} \right) y \sqrt{|y|^2+1}^{-1}, \tag{2.52}$$

then for each δ fixed

$$|r_x|, |r_y| \le o_R(1), \quad ||D^2 \psi_R||_\infty \le o_R(1), \tag{2.53}$$

where $o_R(1)$ means that $\lim_{R\to+\infty} o_R(1) = 0$.

We remark that in the rest of the proof we denote by $o_R(1)$ any function such that $o_R(1) \to 0$ as $R \to +\infty$. We also denote

$$s = C_\delta \alpha |x-y|^{\alpha-2}(x-y). \tag{2.54}$$

Note that the function in (2.43) is smooth near (x,y) by (2.49). Then, since w_δ is a viscosity solution of (2.20) and since (x,y) is a maximum point of the function in (2.43), we have

$$0 \le \mathrm{tr}(\tau(x)\tau(x)^T D^2 w_\delta(x)) - \mathrm{tr}(\tau(y)\tau(y)^T D^2 w_\delta(y)) + L(x,y) + G(x,y) + E(x,y) + F(x,y) + D(x,y) + o_R(1), \tag{2.55}$$

where we used (2.53) to estimate the ψ_R-terms and we denoted

$$D(x,y) = \delta w_\delta(y) - \delta w_\delta(x);$$

$$L(x,y) = s \cdot (b(x) - b(y)) + b(y) \cdot r_y + b(x) \cdot r_x;$$
$$G(x,y) = |\tau(x)^T (s - r_x)|^2 - |\tau(y)^T (s + r_y)|^2;$$
$$E(x,y) = 2\tau(x)\sigma(\bar{x},x)^T \bar{p} \cdot (s - r_x) - 2\tau(y)\sigma(\bar{x},y)^T \bar{p} \cdot (s + r_y);$$
$$F(x,y) = |\sigma^T(\bar{x},x)\bar{p}|^2 - |\sigma^T(\bar{x},y)\bar{p}|^2.$$

First we estimate the second order terms in (2.55), by proving the following lemma.

Lemma 2.5.4. *Under the above notations, we have*

$$tr(\tau(x)\tau(x)^T D^2 w_\delta(x)) - tr(\tau(y)\tau(y)^T D^2 w_\delta(y)) \leq KC_\delta \alpha(\alpha-1)|x-y|^{\alpha-2} + KC_\delta \alpha |x-y|^{\alpha+1} + o_R(1), \quad (2.56)$$

where K is a positive constant (depending on θ of (2.4) *and on the Lipschitz constant of τ) and by $o_R(1)$ we mean that* $\lim_{R\to+\infty} o_R(1) = 0$.

Proof. We observe that, for any orthonormal basis e_i, $i = 1, \cdots m$ of $\mathbb{R}^m$, we can write

$$\mathrm{tr}(\tau(x)\tau(x)^T D^2 w_\delta(x)) = \sum_{i=1}^m (\tau(x)\tau(x)^T D^2 w_\delta(x) e_i, e_i) = \sum_{i=1}^m (D^2 w_\delta(x)\tau(x)e_i, \tau(x)e_i). \quad (2.57)$$

Denote $\phi(t) = C_\delta t^\alpha, f(z) = |z|$. By the maximum point property and the second term of (2.53), we get

$$(D^2 w_\delta(x)p, p) - (D^2 w_\delta(y)q, q) \leq \phi'(f(x-y))(D^2 f(x-y)(p-q), (p-q)) + \phi''(f(x-y))(Df(x-y), p-q)^2 + o_R(1) \quad (2.58)$$

for any $p, q \in \mathbb{R}^m$.
Next we remark that $|Df|^2 = 1$ and therefore, by differentiating this identity, we have $D^2 f Df = 0$. By (2.4), we can set

$$e_1 = \frac{\tau(x)^{-1} Df(x-y)}{|\tau(x)^{-1} Df(x-y)|}, \quad \tilde{e}_1 = -\frac{\tau(y)^{-1} Df(x-y)}{|\tau(y)^{-1} Df(x-y)|}.$$

If $e_1, \tilde{e}_1$ are collinear, the we complete the basis with orthogonal unit vectors $e_i = \tilde{e}_i \in e_1^\top$, $2 \leq i \leq m$. Otherwise, in the plane $\mathrm{span}\{e_1, \tilde{e}_1\}$, we consider a rotation $\mathscr{R}$ of angle $\frac{\pi}{2}$ and we define

$$e_2 = \mathscr{R} e_1, \quad \tilde{e}_2 = -\mathscr{R}\tilde{e}_1.$$

Since $\mathrm{span}\{e_1, e_2\}^\top = \mathrm{span}\{\tilde{e}_1, \tilde{e}_2\}^\top$, we can complete the orthonormal basis with unit vectors $e_i = \tilde{e}_i \in \mathrm{span}\{e_1, e_2\}^\top$, $3 \leq i \leq m$.
By (2.4), we have

$$\theta \leq \frac{1}{|\tau(x)^{-1} Df(x-y)|^2} \leq ||\tau||_\infty^2.$$

Define

$$r_1 = \tau(x)e_1 \quad t_1 = \tau(y)\tilde{e}_1.$$

Since $|Df| = 1$ and $D^2 f Df = 0$ and by choosing $p = r_1, q = r_1$ in (2.58), we get

$$\begin{aligned}(D^2 w_\delta(x) r_1, r_1) - (D^2 w_\delta(y) t_1, t_1) &\le \phi''(f(x-y))(Df(x-y), r_1 - t_1)^2 + o_R(1) \\ &= C_\delta \alpha(\alpha-1)|x-y|^{\alpha-2}(Df(x-y), r_1 - t_1)^2 + o_R(1).\end{aligned}$$

Notice that

$$\alpha(\alpha - 1) < 0. \tag{2.59}$$

By (2.4), we have

$$(Df(x-y), r_1 - t_1)^2 = \left(\frac{1}{|\tau(x)^{-1} Df(x-y)|^2} + \frac{1}{|\tau(y)^{-1} Df(x-y)|} \right)^2 \ge 4\theta.$$

Then

$$(D^2 w_\delta(x) r_1, r_1) - (D^2 w_\delta(y) t_1, t_1) \le 4\theta C_\delta \alpha(\alpha-1)|x-y|^{\alpha-2} + o_R(1). \tag{2.60}$$

Therefore in the right hand side we have a very negative term by a double effect, first because we will choose C_δ large but also because, by doing so, $|x-y|$ becomes smaller and smaller and $|x-y|^{\alpha-2}$ larger and larger.

Now we choose in (2.58) for all $i \in \{1, \cdots, m-1\}$

$$p = \tau(x) e_i \quad q = \tau(y) \tilde{e}_i.$$

Since τ is Lipschitz, we get

$$(D^2 w_\delta(x) \tau(x) e_i, \tau(x) e_i) - (D^2 w_\delta(y) \tau(y) \tilde{e}_i, \tau(y) \tilde{e}_i) \le K C_\delta \alpha |x-y|^{\alpha+1} + o_R(1),$$

where K depends on the Lipschits constant of τ. Then, by summing the previous equation on i and adding (2.60), we get

$$\begin{aligned}\sum_{i=1}^m (D^2 w_\delta(x) \tau(x) e_i, \tau(x) e_i) - \sum_{i=1}^m (D^2 w_\delta(y) \tau(y) \tilde{e}_i, \tau \tilde{e}_i) &\le K C_\delta \alpha(\alpha-1)|x-y|^{\alpha-2} \\ &\quad + K C_\delta \alpha |x-y|^{\alpha+1} + o_R(1),\end{aligned}$$

when by K we denote a constant depending on the Lipschitz constant of τ and on θ. Then, by (2.57) with e_i defined as above (and $\tilde{e}_i$ for $\mathrm{tr}(\tau(y)\tau(y)^T D^2 w_\delta(y))$), we finally get (2.56). □

Then (2.55) becomes

$$\begin{aligned}0 \le K C_\delta \alpha(\alpha-1)|x-y|^{\alpha-2} + K C_\delta \alpha |x-y|^{\alpha+1} + L(x,y) + G(x,y) \\ + E(x,y) + F(x,y) + D(x,y) + o_R(1),\end{aligned} \tag{2.61}$$

Finally we estimate the left terms D, L, G, E, F in (2.61). First note that

$$D(x,y) = \delta w_\delta(y) - \delta w_\delta(x) \leq 0;$$

First note that, by (2.54) and since b is Lipschitz, we have

$$L(x,y) \leq K_1 C_\delta \alpha |x-y|^\alpha + b(y) \cdot r_y + b(x) \cdot r_x.$$

where K_1 depends on the Lipschitz constant of b. Note that

$$b(y) \cdot r_y + b(x) \cdot r_x \leq o_R(1).$$

Indeed, the previous inequality holds from the second of (2.53) when x, y are uniformly bounded in R. Now suppose $|x| \to +\infty$ as $R \to +\infty$ (the argument being similar if $|y| \to +\infty$). By assumption (U) we have

$$b(x) \cdot r_x = (\mu - x) \cdot r_x$$

and by (2.51), we have

$$x \cdot r_x = 2R^{-1}|x|^2 \psi' \left(\frac{\sqrt{|x|^2+1}}{R} \right) (\sqrt{|x|^2+1})^{-1}$$

and since $\psi' \geq 0$ by definition of ψ_R we have

$$x \cdot r_x \geq 0. \tag{2.62}$$

Then by (2.62) and (2.53), we get

$$(\mu - x) \cdot r_x \leq o_R(1).$$

Then

$$L(x,y) \leq K_1 C_\delta \alpha |x-y|^\alpha + o_R(1).$$

Now we estimate the G-term. By the first of (2.53), (2.54) and since τ is bounded, we have

$$G(x,y) \leq |\tau^T(x)s|^2 - |\tau^T(y)s|^2 + K_2 o_R(1) C_\delta \alpha |x-y|^{\alpha-1} + o_R(1),$$

where K_2 depends on $||\tau||_\infty$. Note that from now on we denote by K_2 a constant depending on $||\tau||_\infty$ and the Lipschitz constant of τ and which may change from line to line. Since τ is bounded by (2.54), we have

$$|\tau^T(x)s| + |\tau^T(y)s| \leq K_2 C_\delta \alpha |x-y|^{\alpha-1}$$

and since τ is Lipschitz and by (2.54), we have

$$|\tau^T(x)s| - |\tau^T(y)s| \leq K_2 C_\delta \alpha |x-y|^\alpha.$$

Then we get

$$|\tau^T(x)s|^2 - |\tau^T(y)s|^2 \leq K_2 \alpha C_\delta^2 \alpha |x-y|^{2\alpha-1},$$

and we conclude

$$G(x,y) \leq K_2 \alpha C_\delta^2 |x-y|^{2\alpha-1} + K_2 o_R(1) C_\delta \alpha |x-y|^{\alpha-1} + o_R(1). \tag{2.63}$$

Next we estimate E using the boundedness of σ and we get

$$E(x,y) \leq K_3 \alpha C_\delta |x-y|^{\alpha-1} + o_R(1),$$

where $K_3 > 0$ depends on $\bar{p}, ||\tau||_\infty, ||\sigma||_\infty$.

Finally, by the Lipschitz continuity and boundedness of σ, we have

$$F(x,y) \leq K_4 |x-y|,$$

where K_4 depends on $||\sigma||_\infty$ and the Lipschitz constant of σ and on $\bar{p}$.

Then, by all the previous estimates, (2.61) becomes

$$\begin{aligned} 0 \leq K C_\delta \alpha(\alpha-1)|x-y|^{\alpha-2} + K C_\delta \alpha |x-y|^{\alpha+1} + K_1 C_\delta \alpha |x-y|^\alpha + K_2 \alpha C_\delta^2 |x-y|^{2\alpha-1} \\ + K_2 o_R(1) C_\delta \alpha |x-y|^{\alpha-1} + K_3 \alpha C_\delta |x-y|^{\alpha-1} + K_4 |x-y| + o_R(1). \end{aligned} \tag{2.64}$$

This concludes the proof of Lemma 2.5.3. □

We divide (2.64) by $C_\delta |x-y|^{\alpha-2}$ and we get

$$\begin{aligned} 0 \leq K\alpha(\alpha-1) + K\alpha|x-y|^3 + K_1 \alpha |x-y|^2 + K_2 \alpha C_\delta |x-y|^{\alpha+1} + K_2 o_R(1) \alpha |x-y| \\ + K_3 \alpha |x-y| + K_4 C_\delta^{-1} |x-y|^{3-\alpha} + o_R(1) C_\delta^{-1} |x-y|^{2-\alpha}. \end{aligned} \tag{2.65}$$

Note that by (2.50), we have

$$|x-y| \leq A_\delta^{\frac{1}{\alpha}} C_\delta^{-\frac{1}{\alpha}}, \tag{2.66}$$

then

$$C_\delta^{-1} |x-y|^{3-\alpha} \leq A_\delta^{\frac{3-\alpha}{\alpha}} C_\delta^{-\frac{3}{\alpha}};$$

$$C_\delta^{-1} |x-y|^{2-\alpha} \leq A_\delta^{\frac{2-\alpha}{\alpha}} C_\delta^{-\frac{2}{\alpha}};$$

$$C_\delta |x-y|^{\alpha+1} \leq A_\delta^{\frac{\alpha+1}{\alpha}} C_\delta^{-\frac{1}{\alpha}}.$$

By all the previous estimates and by taking R large enough such that $o_R(1) \le 1$, (2.65) becomes

$$0 \le K\alpha(\alpha-1) + K\alpha A_\delta^{\frac{3}{\alpha}} C_\delta^{-\frac{3}{\alpha}} + K_1\alpha A_\delta^{\frac{2}{\alpha}} C_\delta^{-\frac{2}{\alpha}} + K_2\alpha A_\delta^{\frac{\alpha+1}{\alpha}} C_\delta^{-\frac{1}{\alpha}} + K_2 o_R(1)\alpha A_\delta^{\frac{1}{\alpha}} C_\delta^{-\frac{1}{\alpha}}$$
$$+ K_3\alpha A_\delta^{\frac{1}{\alpha}} C_\delta^{-\frac{1}{\alpha}} + K_4 A_\delta^{\frac{3-\alpha}{\alpha}} C_\delta^{-\frac{3}{\alpha}} + o_R(1) A_\delta^{\frac{2-\alpha}{\alpha}} C_\delta^{-\frac{2}{\alpha}}. \quad (2.67)$$

Then, the claim of the proposition follows by taking C_δ in (2.42) large enough in order to get a contradiction with (2.67). For example we take $C_\delta > \bar{C}_\delta$ where $\bar{C}_\delta$ satisfies

$$K\alpha(\alpha-1) + K\alpha A_\delta^{\frac{3}{\alpha}} \bar{C}_\delta^{-\frac{3}{\alpha}} + K_1\alpha A_\delta^{\frac{2}{\alpha}} \bar{C}_\delta^{-\frac{2}{\alpha}} + K_2\alpha A_\delta^{\frac{\alpha+1}{\alpha}} \bar{C}_\delta^{-\frac{1}{\alpha}} + K_2 o_R(1)\alpha A_\delta^{\frac{1}{\alpha}} \bar{C}_\delta^{-\frac{1}{\alpha}}$$
$$+ K_3\alpha A_\delta^{\frac{1}{\alpha}} \bar{C}_\delta^{-\frac{1}{\alpha}} + K_4 A_\delta^{\frac{3-\alpha}{\alpha}} \bar{C}_\delta^{-\frac{3}{\alpha}} + o_R(1) A_\delta^{\frac{2-\alpha}{\alpha}} \bar{C}_\delta^{-\frac{2}{\alpha}} < 0.$$

Note that $\bar{C}_\delta$ depends on K_i, $i = 1,2,3,4$ and on δ, α, K.

□

Global uniform Lipschitz bound

We prove the following global uniform gradient bound for the solution of the approximate cell problem (2.20).

Proposition 2.5.5. *Let assumptions (U) and (S) hold. Let $w_\delta \in C^2(\mathbb{R}^m)$ be the solution of* (2.20) *(satisfying the bound* (2.22)*). Then for all $x, y \in \mathbb{R}^m$ we have*

$$|w_\delta(y;\bar{x},\bar{p}) - w_\delta(x;\bar{x},\bar{p})| \le C|x-y|, \quad (2.68)$$

where C is a positive constant, depending on $\bar{x}, \bar{p}, ||\tau||_\infty, ||\sigma||_\infty, m$, the Lipschitz constants of τ, b, σ and is independent of δ.

As a straightforward corollary of Proposition 2.5.5, we get the following global gradient bound for the solutions of the cell problem (2.16) in the critical case.

Proposition 2.5.6. *Let assumptions (U) and (S) hold. Let $w \in C^2(\mathbb{R}^m)$ be a solution of* (2.16) *for $\lambda = \bar{H}(\bar{x},\bar{p})$ where $\bar{H}(\bar{x},\bar{p})$ is defined in Proposition 2.3.3. Then*

$$\sup_{y\in\mathbb{R}^m} |D_y w(y;\bar{x},\bar{p})| \le C, \quad (2.69)$$

where C is a positive constant, depending on $\bar{x}, \bar{p}, ||\tau||_\infty, ||\sigma||_\infty, m$. and the Lipschitz constants of τ, b, σ.

Proof of Proposition 2.5.5. Note that, under the assumption (U), (2.20) reads for $|y| > R_1$

$$\delta w_\delta + F(\bar{x}, y, \bar{p}, Dw_\delta, D^2 w_\delta) - |\sigma(\bar{x},y)\bar{p}|^2 = 0, \quad (2.70)$$

where

$$F(\bar{x},y,\bar{p},q,Y) := -\mathrm{tr}(\tau\tau^T Y) - |\tau^T q|^2 - (\mu - y, q) - (2\tau\sigma^T(\bar{x},y)\bar{p}, q).$$

Note also that throughout the following proof we denote either by (a,b) or $a \cdot b$ the scalar product for any $a,b \in \mathbb{R}^m$.

Let $\bar{R} > R_1$ be large enough (which will be chosen suitably at the end of the proof) and take $C_{\bar{R}}$ the constant of Lemma 2.5.1 for $k = \bar{R}$. Then we have for all $x,y \in \bar{B}_{\bar{R}}$

$$|w_\delta(x;\bar{x},\bar{p}) - w_\delta(y;\bar{x},\bar{p})| \leq C_{\bar{R}}|x-y|, \tag{2.71}$$

where we included the dependence on $\bar{p}$ into $C_{\bar{R}}$ for simplicity. For convenience of notation in the following we drop the dependence on $\bar{x},\bar{p}$ by denoting the solution of (2.70) by w_δ.

In this first part of the proof we proceed analogously as in the proof of Proposition 2.5.2. The new part of the proof starts from Lemma 2.5.7. We give a sketch and for all the details we refer to the beginning of the proof of Proposition 2.5.2.

We proceed by contradiction and we suppose that

$$\sup\{w_\delta(x) - w_\delta(y) - C|x-y|\} = M > 0, \tag{2.72}$$

where C is a positive constant large enough, that is $C > \max\{C_{\bar{R}}, C_{\bar{R}+1}\}$.

Let $R > 0$ and consider the function

$$\Phi(x,y) = w_\delta(x) - w_\delta(y) - C|x-y| - \psi_R(x) - \psi_R(y), \tag{2.73}$$

where

$$\psi_R(z) = \psi\left(\frac{\sqrt{|z|^2+1}}{R}\right) \tag{2.74}$$

where ψ is defined in (5.43). By standard argument (see also the proof of Proposition 2.5.2), we prove that

$$M_R = \sup \Phi(x,y) \to M \text{ as } R \to +\infty,$$

then we can suppose for R large enough

$$M_R \geq \frac{M}{2} > 0, \tag{2.75}$$

and by definition of ψ_R we get that, for R large enough, there exist (x_R, y_R) such that

$$M_R = w_\delta(x_R) - w_\delta(y_R) - C|x_R - y_R| - \psi_R(x_R) - \psi_R(y_R). \tag{2.76}$$

Note also that

$$|x_R - y_R| > 0. \tag{2.77}$$

We prove the following lemma.

Lemma 2.5.7. *Under the above notations, we have that, for R large enough, there exists a point of maximum (x_R, y_R) of the function Φ such that $(x_R, y_R) \in (\mathbb{R}^m \setminus \bar{B}_{\bar{R}}) \times (\mathbb{R}^m \setminus \bar{B}_{\bar{R}})$. Moreover*

$$\liminf_{R \to +\infty} |x_R - y_R| > 0. \tag{2.78}$$

Remark 2.5.8. The result of Lemma 2.5.7 is essential in order to use assumption (U) in the rest of the proof. Indeed, the radius $\bar{R}$ is chosen such that $\bar{R} > R_1$ (R_1 being defined in assumption (U)).

Remark 2.5.9. As we will show throughout the proof, we note that a key result in order to prove (2.78) is the Hölder bound of Proposition 2.5.2.

Proof. Let (x_R, y_R) be a point of maximum of Φ defined in (2.73) (see the above arguments for the existence). If $(x_R, y_R) \in (\mathbb{R}^m \setminus \bar{B}_{\bar{R}}) \times (\mathbb{R}^m \setminus \bar{B}_{\bar{R}})$, the claim is proved. Otherwise, there are three possible cases (up to subsequences):

(i) $(x_R, y_R) \in \bar{B}_{\bar{R}} \times \bar{B}_{\bar{R}}$;
(ii) $(x_R, y_R) \in \bar{B}_{\bar{R}} \times (\mathbb{R}^m \setminus \bar{B}_{\bar{R}})$;
(iii) $(x_R, y_R) \in (\mathbb{R}^m \setminus \bar{B}_{\bar{R}}) \times \bar{B}_{\bar{R}}$.

Suppose we are in case (i). We apply the local estimate on $\bar{B}_{\bar{R}}$ (2.71) and by the choice of C in (2.72), we get a contradiction with (2.75).

Now we deal with case (ii) and we observe that case (iii) can be treated analogously. We prove that there exists $z_R \in \mathbb{R}^m \setminus \bar{B}_{\bar{R}}$ such that (z_R, y_R) is still a maximum point of the function Φ. Note that we can suppose that $y_R \in \mathbb{R}^m \setminus \bar{B}_{\bar{R}+1}$. Indeed, if $y_R \in \bar{B}_{\bar{R}+1}$, we use the local estimate on $\bar{B}_{\bar{R}+1}$ and by the choice of C in (2.72), we get a contradiction with (2.75). Let z_R, z'_R be respectively the points where the segment between x_R and y_R intersects the boundary of $B_{\bar{R}+1}$ and of $B_{\bar{R}}$. Note that

$$|x_R - y_R| = |x_R - z_R| + |z_R - y_R| \tag{2.79}$$

and

$$|x_R - z_R| = |x_R - z'_R| + 1. \tag{2.80}$$

Then, by (2.79), we have

$$\max \Phi = \Phi(x_R, y_R) \leq \Phi(z_R, y_R) + w_\delta(x_R) - w_\delta(z_R) - C|x_R - z_R| - \psi_R(x_R) + \psi_R(z_R),$$

and by the local estimate (2.71) on $\bar{B}_{\bar{R}+1}$ coupled with (2.80), we get

$$\max\Phi \leq \Phi(z_R,y_R) + C_{\bar{R}+1}|x_R - z'_R| + C_{\bar{R}+1} - C|x_R - z'_R| - C - \psi_R(x_R) + \psi_R(z_R).$$

By the choice of C in (2.72) we get

$$\max\Phi \leq C_{\bar{R}+1} - C + \Phi(z_R,y_R) - \psi_R(x_R) + \psi_R(z_R)$$

and, by taking R large enough so that $C_{\bar{R}+1} - C - \psi_R(x_R) + \psi_R(z_R) \leq 0$, we conclude

$$\max\Phi \leq \Phi(z_R,y_R).$$

Then, for R large enough, $(z_R,y_R) \in (\mathbb{R}^m \setminus \bar{B}_{\bar{R}}) \times (\mathbb{R}^m \setminus \bar{B}_{\bar{R}})$ is a point of maximum of the function Φ. This conclude the proof of the first claim.

Now we prove (2.78). By contradiction, we suppose that

$$\liminf_{R\to+\infty}|x_R - y_R| = 0.$$

By (2.76) and the definition of ψ_R, we have

$$M_R \leq w_\delta(x_R) - w_\delta(y_R).$$

Now we use Proposition 2.5.2 and by (2.41), we get

$$M_R \leq C_\delta|x_R - y_R|^\alpha.$$

Then, since $M_R \to M > 0$, we get the following contradiction

$$0 < \liminf_{R\to+\infty} M_R \leq \liminf_{R\to+\infty} C_\delta|x_R - y_R|^\alpha = 0,$$

concluding the proof. □

From now on we omit the dependence on R and we write

$$(x_R,y_R) = (x,y).$$

We prove the following lemma.

Lemma 2.5.10. *Under the above notations and assumptions, there exists two positive constants K_1, K_2 such that*

$$C|x-y| \leq CK_1 g(x,y)|x-y| + K_2|x-y| + o_R(1), \tag{2.81}$$

where $g : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^+$ is such that $\forall \varepsilon > 0$ there exists R_ε such that $g(x,y) \le \varepsilon$ for all $|x|, |y| \ge R_\varepsilon$. Moreover K_1, K_2 depends only on $\bar{p}, ||\sigma||_\infty, ||\tau||_\infty$ and by $o_R(1)$ we mean that $\lim_{R\to+\infty} o_R(1) = 0$.

Remark 2.5.11. Note that $C|x-y|$, on the left side in (2.81), remains striclty positive for $R \to +\infty$ (by Lemma (2.5.7) proved in step 2). This term stems from the Ornstein-Uhlenbeck term $-(\mu - y)\cdot Dw_\delta$ in the cell problem (2.70).

Proof. We denote

$$r_x := D\psi_R(x) = R^{-1}\psi'\left(\frac{\sqrt{|x|^2+1}}{R}\right) x(\sqrt{|x|^2+1})^{-1} \tag{2.82}$$

$$r_y := D\psi_R(y) = R^{-1}\psi'\left(\frac{\sqrt{|y|^2+1}}{R}\right) y(\sqrt{|y|^2+1})^{-1}. \tag{2.83}$$

We remark that

$$|r_x|, |r_y| \le R^{-1}||\psi'||_\infty, \tag{2.84}$$

where $||\psi'||_\infty$ depends on δ. Similarly we argue for the second derivatives of ψ_R and we get

$$||D^2\psi_R(z)||_\infty \le o_R(1), \tag{2.85}$$

where $o_R(1)$ means that $\lim_{R\to+\infty} o_R(1) = 0$.

Note that in the rest of the proof we denote by $o_R(1)$ any function respectively such that $o_R(1) \to 0$ as $R \to +\infty$. We also denote

$$s = C\frac{x-y}{|x-y|}. \tag{2.86}$$

Notice that the function in (2.73) is smooth since for R big enough $x \ne y$ by Lemma 2.5.7. Then, since w_δ is a viscosity solution of (2.70) and since (x,y) is a maximum point of the function in (2.73), we have

$$\begin{aligned} L(x,y) \le \operatorname{tr}(\tau\tau^T D^2 w_\delta(x)) - \operatorname{tr}(\tau\tau^T D^2 w_\delta(y)) + o_R(1) + G(x,y) \\ + E(x,y) + F(x,y) + D(x,y), \end{aligned} \tag{2.87}$$

where we used (2.84) and (2.85) to estimate the ψ_R-terms and where we denote

$$D(x,y) = \delta w_\delta(y) - \delta w_\delta(x);$$

$$L(x,y) = (s,(x-y)) - (\mu - y, r_y) - (\mu - x, r_x);$$

$$G(x,y) = |\tau^T(s+r_x)|^2 - |\tau^T(s-r_y)|^2;$$
$$E(x,y) = (2\tau\sigma(\bar{x},x)^T\bar{p}, s+r_x) - (2\tau\sigma(\bar{x},y)^T\bar{p}, s-r_y);$$
$$F(x,y) = |\sigma^T(\bar{x},x)\bar{p}|^2 - |\sigma^T(\bar{x},y)\bar{p}|^2.$$

We estimate each term in (2.87). The most important terms is L since it gives rise to the left order term $C|x-y|$ in (2.81). Indeed by (2.86), we have

$$L(x,y) \geq C|x-y| - (\mu - y)\cdot r_y - (\mu - x)\cdot r_x$$

and notice that by (2.82) and (2.83) we have

$$x \cdot r_x = R^{-1}|x|^2\psi'\left(\frac{\sqrt{|x|^2+1}}{R}\right)(\sqrt{|x|^2+1})^{-1}$$

and

$$y \cdot r_y = R^{-1}|y|^2\psi'\left(\frac{\sqrt{|y|^2+1}}{R}\right)y(\sqrt{|y|^2+1})^{-1}$$

and since $\psi' \geq 0$ by definition of ψ_R, we have

$$x \cdot r_x \geq 0, \quad y \cdot r_y \geq 0. \tag{2.88}$$

By (2.88) and (2.84), we get

$$-(\mu - y)\cdot r_y - (\mu - x)\cdot r_x \geq o_R(1),$$

and then

$$L(x,y) \geq C|x-y| + o_R(1).$$

Then by the previous estimates we get

$$C|x-y| \leq \mathrm{tr}(\tau\tau^T D^2 w_\delta(x)) - \mathrm{tr}(\tau\tau^T D^2 w_\delta(y)) + o_R(1) + G(x,y) + E(x,y) + F(x,y) + D(x,y), \tag{2.89}$$

Now we estimate the remaining terms in the right-hand side of (2.89). First note that

$$D(x,y) = \delta w_\delta(y) - \delta w_\delta(x) \leq 0.$$

By (2.84) and (2.86), we have

$$G(x,y) \leq o_R(1). \tag{2.90}$$

Next, by (S) and the boundedness of σ, we have

$$E(x,y) \leq CK_1 g(x,y)|x-y| + o_R(1),$$

where $K_1 > 0$ depends on $\bar{p}, ||\tau||_\infty, ||\sigma||_\infty$.

By the Lipschitz continuity and boundedness of σ, we have

$$F(x,y) \leq K_2|x-y|,$$

where K_2 depends on $||\sigma||_\infty$ and the Lipschitz constant of σ and on $\bar{p}$.

Finally we estimate the second order terms in (2.89) as follows

$$\mathrm{tr}(\tau\tau^T D^2 w_\delta(x)) - \mathrm{tr}(\tau\tau^T D^2 w_\delta(y)) \leq o_R(1). \tag{2.91}$$

where by $o_R(1)$ we mean that $\lim_{R\to+\infty} o_R(1) = 0$. The proof of (2.91) is analogous to the proof of Lemma 2.5.4, (2.56) proven in Proposition 2.5.2 and even simpler. Indeed, we use again the following property: if e_i, $i = 1, \cdots m$ is an orthonormal basis of $\mathbb{R}^m$ and A is a matrix $m \times m$, we have

$$\mathrm{tr}(A) = \sum_{i=1}^{m} (Ae_i, e_i),$$

then for any orthonormal basis e_i, $i = 1, \cdots m$ of $\mathbb{R}^m$, we can write

$$\mathrm{tr}(\tau\tau^T D^2 w_\delta(x)) = \sum_{i=1}^{m} (\tau\tau^T D^2 w_\delta(x) e_i, e_i) = \sum_{i=1}^{m} (D^2 w_\delta(x)\tau e_i, \tau e_i). \tag{2.92}$$

Denote $f(z) = |z|$. We recall that the function in (2.73) is smooth at $(x,y) = (x_R, y_R)$ for R large enough by Lemma 2.5.7. Then, since x,y is a maximum point of the function in (2.73) and by (2.85), we get

$$(D^2 w_\delta(x)p, p) - (D^2 w_\delta(y)q, q) \leq C(D^2 f(x-y)(p-q), (p-q)) + o_R(1) \tag{2.93}$$

for any $p, q \in \mathbb{R}^m$. Then, in order to prove the claim, it is enough to choose in (2.93) for all $i \in \{1, \cdots m\}$

$$p = \tau e_i, \quad q = \tau e_i.$$

Then we get

$$(D^2 w_\delta(x)\tau e_i, \tau e_i) - (D^2 w_\delta(y)\tau e_i, \tau e_i) \leq o_R(1) \;\text{ for all } i \in \{1, \cdots m\},$$

and by summing the previous equation on i, we get

$$\sum_{i=1}^{m}(D^2 w_\delta(x)\tau e_i, \tau e_i) - \sum_{i=1}^{m}(D^2 w_\delta(y)\tau e_i, \tau e_i) \le o_R(1)$$

from which we conclude (2.91). By coupling all the previous estimates, we get (2.81) and we conclude the proof of Lemma 2.5.10. $\square$

Now we conclude the the argument as follows. We use assumption (S) and by taking $\bar{R} > R_1$ large enough, we consider $|x|, |y|$ large enough, such that

$$K_1 g(x,y) \le \frac{1}{2}. \tag{2.94}$$

Now we send $R \to +\infty$ in (2.81) and divide by $|x-y|$ thanks to Lemma 2.5.7, and we get

$$C \le \frac{C}{2} + K_2, \tag{2.95}$$

Then, to get a contradiction with (2.95), it is enough to take C large enough such that

$$C > 2K_2. \tag{2.96}$$

Note that $C, \bar{R}$ depend respectively only on K_2, R_1 and in particular, they are independent on δ.

Then the proof follows by taking C in (2.72), such that $C > \max\{C_{\bar{R}}, C_{\bar{R}+1}, 2K_2\}$, where $\bar{R} > R_1$ is such that (2.94) holds (and we recall that $C_{\bar{R}} = C'_{\bar{R}}(1+|\bar{p}|), C_{\bar{R}+1} = C'_{\bar{R}+1}(1+|\bar{p}|)$, where $C'_{\bar{R}}, C'_{\bar{R}+1}$ are the constants of Lemma 2.5.1 for $k = \bar{R}, \bar{R}+1$ respectively.)

$\square$

2.5.2 Supercritical case

In the supercritical case, we proceed analogously as in the critical case. We prove first a Hölder bound, not uniform in δ, for the solution of the δ-cell problem (2.37) (see Proposition 2.5.12). This result is analogous to Proposition 2.5.2. Then, we prove the global uniform bound in Proposition 2.5.14.

In this case we do not prove a local gradient bound (as we do in the critical case, Lemma 2.5.1). Indeed, in the supercritical case the δ-cell problem is the following

$$\delta w_\delta(y) - \mathrm{tr}(\tau\tau^T(y)D^2 w_\delta(y)) - b(y)\cdot Dw_\delta(y) - |\sigma^T(\bar{x},y)\bar{p}|^2 = 0, \tag{2.97}$$

which is linear in the gradient. Since the equation is no more coercive in the gradient, we do not expect to prove by the Bernstein method a local gradient bound analogous to Lemma

2.5.1. We remark also that by the Ishii-Lions we are able to obtain local estimates, but depending on the L^∞-norm of the solution, therefore on δ.

For this reason in the supercritical case we strengthen assumption (U) on the process Y_t and we consider the Ornstein-Uhlenbeck process in all the space. Note also that, in this case, there is no need of assumption (S), since (S) is used only to estimate the correlation term $\tau^T \sigma^T \bar{p} \cdot Dw_\delta$, which is not present in the cell problem for $\alpha > 2$. Once (U) holds in all the space, the proofs are exactly the same and even easier.

Global Hölder bound

As in the critical case, we prove first the following Hölder bound. We just sketch the proof and for all the details we refer to the proof of Proposition 2.5.2.

Proposition 2.5.12. *Let assumption (U) holds. Let $w_\delta(\cdot;\bar{x},\bar{p}) \in C^2(\mathbb{R}^m)$ be a solution of* (2.37). *Then there exists $C_\delta > 0$ and $\alpha \in (0,1)$ such that*

$$|w_\delta(x;\bar{x},\bar{p}) - w_\delta(y;\bar{x},\bar{p})| \le C_\delta |x-y|^\alpha \quad \textit{for all } x,y \in \mathbb{R}^m, \tag{2.98}$$

where C_δ depends on $\delta, \alpha, ||\tau||_\infty, ||\sigma(\bar{x},\cdot)||_\infty, \bar{p}$ and the Lipschitz constant of σ.

Proof. We omit the dependence on $\bar{x},\bar{p}$ and we denote $w_\delta(x;\bar{x},\bar{p}) = w_\delta(x)$.

Let $\delta > 0$ and $\alpha \in (0,1)$ let $C_\delta > 0$ be large enough. Note that C_δ will be choosen suitably at the end of the proof and will depend on $\delta, \alpha, ||\tau||_\infty, ||\sigma(\bar{x},\cdot)||_\infty, \bar{p}$, the lipschitz constant of σ. For clearness of exposition, we keep track only of the dependence on δ. We suppose that

$$\sup\{w_\delta(x) - w_\delta(y) - C_\delta |x-y|^\alpha\} = M > 0.$$

Let $R > 0$ and consider the function

$$\Phi(x,y) = w_\delta(x) - w_\delta(y) - C_\delta |x-y|^\alpha - \psi_R(x) - \psi_R(y),$$

where $\psi_R(z) = \psi\left(\frac{\sqrt{|z|^2+1}}{R}\right)$ and $\psi \in C^2([0,+\infty))$ is defined in (5.43). By standard arguments (and as already shown in Proposition 2.5.2) there exists (x_R, y_R) point of maximum of Φ such that $|x_R - y_R| > 0$, and

$$|x_R - y_R| \le \left(\frac{A_\delta}{C_\delta}\right)^{\frac{1}{\alpha}}. \tag{2.99}$$

From now on we omit the dependence on R and we write

$$(x_R, y_R) = (x,y).$$

The main result is the following lemma.

Lemma 2.5.13. *Under the above notations, there exists positive constants K, K_1, K_2 such that*

$$0 \leq KC_\delta \alpha(\alpha-1)|x-y|^{\alpha-2} + C_\delta \alpha |x-y|^\alpha + K_2|x-y| + o_R(1) \tag{2.100}$$

where $o_R(1)$ means that $o_R(1) \to 0$ as $R \to +\infty$. Moreover K, K_2 depends only on $\bar{p}, ||\sigma||_\infty, ||\tau||_\infty$, the determinant of τ and the Lipschitz constant of σ.

Proof. If $r_x = D_x \psi_R$ and $r_y = D_y \psi_R$, then for each δ fixed

$$|r_x|, |r_y| \leq o_R(1), \quad ||D^2\psi_R||_\infty \leq o_R(1), \tag{2.101}$$

where $o_R(1)$ means that $\lim_{R\to+\infty} o_R(1) = 0$. We also denote

$$s = C_\delta \alpha |x-y|^{\alpha-2}(x-y). \tag{2.102}$$

Since the function Φ is smooth near (x,y), since w_δ is a viscosity solution of (2.37) and (x,y) is a maximum point of Φ, we have

$$0 \leq \mathrm{tr}(\tau\tau^T D^2 w_\delta(x)) - \mathrm{tr}(\tau\tau^T D^2 w_\delta(y)) + o_R(1) + L(x,y) + F(x,y) + D(x,y), \tag{2.103}$$

where τ is defined in (U) and we estimated the ψ_R-terms by (2.101) and

$$D(x,y) = \delta w_\delta(y) - \delta w_\delta(x) \leq 0;$$

$$F(x,y) = |\sigma^T(\bar{x},x)\bar{p}|^2 - |\sigma^T(\bar{x},y)\bar{p}|^2 \leq K_2|x-y|,$$

where K_2 depends on $||\sigma||_\infty$ and the Lipschitz constant of σ and on $\bar{p}$ and

$$L(x,y) = s\cdot(b(y)-b(x)) + b(y)\cdot r_y + b(x)\cdot r_x.$$

As showed in the proof of Proposition 2.5.5 we have

$$L(x,y) \leq C_\delta \alpha |x-y|^\alpha + o_R(1).$$

Moreover

$$\mathrm{tr}(\tau\tau^T D^2 w_\delta(x)) - \mathrm{tr}(\tau\tau^T D^2 w_\delta(y)) \leq KC_\delta \alpha(\alpha-1)|x-y|^{\alpha-2} + o_R(1),$$

where K is a positive constant (depending on the determinant of τ and on $||\tau||_\infty$).

Then, by all the previous estimates, we get (2.100), concluding the proof of Lemma 2.5.13. □

We divide (2.100) by $C_\delta|x-y|^{\alpha-2}$ and we get

$$0 \le K\alpha(\alpha-1)+\alpha|x-y|^2+K_2C_\delta^{-1}|x-y|^{3-\alpha}+o_R(1)C_\delta^{-1}|x-y|^{2-\alpha}. \tag{2.104}$$

Note that by (2.99), we have

$$|x-y| \le A_\delta^{\frac{1}{\alpha}}C_\delta^{-\frac{1}{\alpha}}, \quad C_\delta^{-1}|x-y|^{3-\alpha} \le A_\delta^{\frac{3-\alpha}{\alpha}}C_\delta^{-\frac{3}{\alpha}}, \quad C_\delta^{-1}|x-y|^{2-\alpha} \le A_\delta^{\frac{2-\alpha}{\alpha}}C_\delta^{-\frac{2}{\alpha}}.$$

By all the previous estimates and by taking R large enough such that $o_R(1) \le 1$, (2.104) becomes

$$0 \le K\alpha(\alpha-1)+\alpha A_\delta^{\frac{2}{\alpha}}C_\delta^{-\frac{2}{\alpha}}+K_2A_\delta^{\frac{3-\alpha}{\alpha}}C_\delta^{-\frac{3}{\alpha}}+A_\delta^{\frac{2-\alpha}{\alpha}}C_\delta^{-\frac{2}{\alpha}}. \tag{2.105}$$

Then, the claim of the proposition follows by taking C_δ large enough in order to get a contradiction with (2.105). For example we take $C_\delta > \bar{C}_\delta$ where $\bar{C}_\delta$ satisfies

$$K\alpha(\alpha-1)+\alpha A_\delta^{\frac{2}{\alpha}}\bar{C}_\delta^{-\frac{2}{\alpha}}+K_2A_\delta^{\frac{3-\alpha}{\alpha}}\bar{C}_\delta^{-\frac{3}{\alpha}}+A_\delta^{\frac{2-\alpha}{\alpha}}\bar{C}_\delta^{-\frac{2}{\alpha}} < 0.$$

Note that $\bar{C}_\delta$ depends only on K_i, $i=1,2$ and on δ,α,K.

□

Global uniform Lipschitz bound

We prove the following uniform gradient bound for the solution of the approximate cell problem (2.37), see Proposition 2.5.14. The proof is very similar (and even easier) to the proof of Proposition 2.5.5. As already noted, in this case we proceed directly to the estimate in all the space without proving first the local uniform bound of Lemma 2.5.1. The global estimate is achieved mainly thanks to assumption (U), which in the supercritical case holds in all the space. Note also that in this case in the cell problem (2.37) there is no more the correlation term $\tau(y)\sigma^T(\bar{x},y)\bar{p}\cdot Dw_\delta$, therefore in this case we do not need assumption (S) to prove the result.

Proposition 2.5.14. *Let assumptions (U) hold. Let $w_\delta \in C^2(\mathbb{R}^m)$ be the solution of* (2.37) *(satisfying the bound* (2.38)*),. Then for all $x,y \in \mathbb{R}^m$ we have*

$$|w_\delta(y;\bar{x},\bar{p})-w_\delta(x;\bar{x},\bar{p})| \le C|x-y|, \tag{2.106}$$

where C is a positive constant, depending on $\bar{x},\bar{p}$, the Lipschitz constant of σ and independent of δ.

As a straightforward consequence we get the following global gradient bound for any solution of (2.35).

Proposition 2.5.15. *Let assumption (U) hold. Let $w \in C^2(\mathbb{R}^m)$ be a solution of* (2.35) *for $\lambda = \bar{H}(\bar{x}, \bar{p})$ where $\bar{H}(\bar{x}, \bar{p})$ is defined in Proposition 2.4.3. Then*

$$\sup_{y \in \mathbb{R}^m} |D_y w(y; \bar{x}, \bar{p})| \le C, \tag{2.107}$$

where C is a positive constant, depending on $\bar{x}, \bar{p}$ and the Lipschitz constant of σ.

Proof of Proposition 2.5.14. We just sketch the proof and for all the details we refer to the proof of Proposition 2.5.5. We drop the dependence on $\bar{x}, \bar{p}$ by denoting the solution of (2.37) by w_δ. By contradiction, we suppose that

$$\sup\{w_\delta(x) - w_\delta(y) - C|x-y|\} > 0,$$

where $C > 0$ is a constant large enough. Let $R > 0$ and consider the function

$$\Phi(x,y) = w_\delta(x) - w_\delta(y) - C|x-y| - \psi_R(x) - \psi_R(y),$$

where C is a positive constant large enough and $\psi_R(z) = \psi\left(\frac{\sqrt{|z|^2+1}}{R}\right)$ where ψ is defined in (5.43). By standard argument (see also the proof of Proposition 2.5.5), we prove that there exist (x_R, y_R) such that point of maximum of Φ and $|x_R - y_R| > 0$.

We first prove the following lemma. We omit the proof since it is the same as the proof of Lemma 2.5.7. Note only that in the place of Proposition 2.5.2 we use Proposition 2.5.12.

Lemma 2.5.16. *Under the above notations, we have that*

$$\liminf_{R \to +\infty} |x_R - y_R| > 0. \tag{2.108}$$

From now on we omit the dependence on R and we write

$$(x_R, y_R) = (x, y).$$

The main result is the following lemma.

Lemma 2.5.17. *Under the above notations and assumptions, there exists a positive constant K_2 such that*

$$C|x-y| \le K_2|x-y| + o_R(1), \tag{2.109}$$

where $o_R(1)$ means that $o_R(1) \to 0$ as $R \to +\infty$ and K_2 depends only on $\bar{p}$ and the Lipschitz constant of σ.

Proof. We denote $r_x := D\psi_R(x), r_y := D\psi_R(y)$ and note that $|r_x|, |r_y| \leq R^{-1}||\psi'||_\infty$, where $||\psi'||_\infty$ depends on δ. Similarly we argue for the second derivatives of ψ_R and we get $||D^2\psi_R(z)||_\infty \leq o_R(1)$, where $o_R(1)$ means that $\lim_{R\to+\infty} o_R(1) = 0$.

We also denote

$$s = C\frac{x-y}{|x-y|}. \tag{2.110}$$

Notice that the function Φ is smooth since for R big enough $x \neq y$ by Lemma 2.5.16. Then, since w_δ is a viscosity solution of (2.37) and since (x,y) is a maximum point of Φ, we have

$$L(x,y) \leq \mathrm{tr}(\tau\tau^T D^2 w_\delta(x)) - \mathrm{tr}(\tau\tau^T D^2 w_\delta(y)) + o_R(1) + F(x,y) + D(x,y), \tag{2.111}$$

where we estimated the ψ_R-terms by $o_R(1)$ and

$$D(x,y) = \delta w_\delta(y) - \delta w_\delta(x) \leq 0;$$

$$L(x,y) = (s,(x-y)) - (\mu - y, r_y) - (\mu - x, r_x) \geq C|x-y| + o_R(1);$$
$$F(x,y) = |\sigma^T(\bar{x},x)\bar{p}|^2 - |\sigma^T(\bar{x},y)\bar{p}|^2 \leq K_2|x-y|$$

where K_2 depends on $||\sigma||_\infty$ and the Lipschitz constant of σ and on $\bar{p}$. For the second order terms we have

$$\mathrm{tr}(\tau\tau^T D^2 w_\delta(x)) - \mathrm{tr}(\tau\tau^T D^2 w_\delta(y)) \leq o_R(1). \tag{2.112}$$

where $o_R(1)$ means that $o_R(1) \to 0$ as $R \to +\infty$. The proof of (2.112) is carried out exactly as in the critical case. Then , by coupling all the previous estimates, we get (2.109) and we conclude the proof of the lemma. □

Now we send $R \to +\infty$ in (2.109) and divide by $|x-y|$ thanks to Lemma 2.5.16, and we get $C \leq K_2$, and we trvially get a contradiction if we take $C > K_2$. Note that C depends only on K_2 and in particular, is independent on δ.

□

2.6 The comparison principle

In this section we provide the comparison principle for the limit PDE

$$v_t - \bar{H}(x,Dv) = 0 \quad \text{in } (0,T)\times\mathbb{R}^n, \tag{2.113}$$

where $\bar{H}$ is defined in Proposition 2.3.3 for $\alpha = 2$ and in Proposition 2.4.3 for $\alpha > 2$.

Note that the comparison principle for the limit problem is a crucial ingredient in the proof of the convergence, which we address in the following section.

Theorem 2.6.1. *Let assumption (U) holds. Let $u \in BUSC([0,T]\times\mathbb{R}^n)$ and $v \in BLSC([0,T]\times\mathbb{R}^n)$ be, respectively, a subsolution and a supersolution to* (2.113) *such that $u(0,x) \le v(0,x)$ for all $x \in \mathbb{R}^n$. Then $u(x,t) \le v(x,t)$ for all $x \in \mathbb{R}^n$ and $0 \le t \le T$.*

Proof. We observe that the effective Hamiltonian $\bar{H}$ satisfies the properties $(a),(b),(c),(d)$ of Proposition 2.3.6. Then the proof is as the same of Theorem 1.2.6. □

2.7 The convergence result

In this section we prove the convergence theorem of v^ε to the unique solution of the limit problem (2.116). Throughout this section, let assumptions (U) and (S) hold. Let $\alpha \ge 2$. We recall that v^ε denotes the unique bounded viscosity solution of

$$\begin{cases} \partial_t v^\varepsilon - H^\varepsilon\left(x,y,D_x v^\varepsilon, \frac{D_y v^\varepsilon}{\varepsilon^{\alpha-1}}, D^2_{xx}v^\varepsilon, \frac{D^2_{yy}v^\varepsilon}{\varepsilon^{\alpha-1}}, \frac{D^2_{xy}v^\varepsilon}{\varepsilon^{\frac{\alpha-1}{2}}}\right) = 0 & \text{in } [0,T]\times\mathbb{R}^n\times\mathbb{R}^m, \\ v^\varepsilon(0,x,y) = h(x) & \text{in } \mathbb{R}^n\times\mathbb{R}^m. \end{cases} \tag{2.114}$$

where

$$\begin{aligned} H^\varepsilon(x,y,p,q,X,Y,Z) \;&:=\; |\sigma^T p|^2 + b\cdot q + \operatorname{tr}(\tau\tau^T Y) + \varepsilon\left(\operatorname{tr}(\sigma\sigma^T X) + \phi\cdot p\right) \\ &+\; 2\varepsilon^{\frac{\alpha}{2}-1}(\tau\sigma^T p)\cdot q + 2\varepsilon^{\frac{1}{2}}\operatorname{tr}(\sigma\tau^T Z) + \varepsilon^{\alpha-2}|\tau^T q|^2. \end{aligned}$$

We state and prove the convergence result. We will make use of the relaxed semi-limits which we define as follows. The lower semi-limit $\underline{v}$ is,

$$\underline{v}(t,x) := \liminf_{\varepsilon\to 0}\{v^\varepsilon(t_\varepsilon,x_\varepsilon,y_\varepsilon)\,|\,x_\varepsilon \to x,\, t_\varepsilon \to t,\, y_\varepsilon \text{ bounded}\}$$

and the upper semi-limit $\bar{v}$ is

$$\bar{v}(t,x) := \limsup_{\varepsilon\to 0}\{v^\varepsilon(t_\varepsilon,x_\varepsilon,y_\varepsilon)\,|\,x_\varepsilon \to x,\, t_\varepsilon \to t,\, y_\varepsilon \text{ bounded}\}.$$

Since h is bounded, the family v^ε is equibounded and we have $\bar{v} \in BUSC([0,T]\times\mathbb{R}^n)$ and $\underline{v} \in BLSC([0,T]\times\mathbb{R}^n)$. Notice that by definition, we have

$$\bar{v}(x,t) \ge \underline{v}(x,t). \tag{2.115}$$

Theorem 2.7.1. *Let assumption (U) holds and for $\alpha = 2$ let assumption (S) holds . Recall the effective problem*

$$v_t - \bar{H}(x,Dv) = 0 \text{ in } (0,T)\times\mathbb{R}^n \quad v(0,x) = h(x) \text{ on } \mathbb{R}^n \tag{2.116}$$

where $\bar{H}$ is defined by Proposition 2.3.3 for $\alpha = 2$ and Proposition 2.4.3 for $\alpha > 2$. Then
a) the upper limit $\bar{v}$ of v^ε is a subsolution of (2.116)*;*
b) the lower limit $\underline{v}$ is a supersolution of (2.116)*;*
c) v^ε converges uniformly on the compact subsets of $[0,T) \times \mathbb{R}^n \times \mathbb{R}^m$ to the unique viscosity solution of (2.116)*.*

Proof. We split the proof into three steps. In Step 1 we prove *a*), in Step 2 we prove *b*) and in Step 3 we prove *c*). Since the proofs of *a*) and *b*) are analogous, in step *b*) we give only the main details.

Step. *1 (Proof of a))* Since the proofs for the critical and supercritical case are analogous with some minor adaptations, we treat first the case $\alpha = 2$ and at the end of the proof in Remark 2.7.4 we describe the major changes for $\alpha > 2$.

We take a smooth function ψ, and without loss of generality we assume that ψ is coercive in the variable x and for all compact $K \subset [0,T] \times \mathbb{R}^n$ there exists a constant $C_K > 0$ such that

$$|\partial_t \psi(t,x)| \leq C_K \quad \forall (t,x) \in K. \tag{2.117}$$

Let $(\bar{t},\bar{x})$ be a point of strict maximum of $\bar{v}(t,x) - \psi(t,x)$. Let $\eta > 0$ and consider the function

$$\Phi(t,x,y) = v^\varepsilon(t,x,y) - \psi(t,x) - \varepsilon(w(y) + \eta\chi(y)), \tag{2.118}$$

where χ is the Liapounov function defined in (2.11) and w is the corrector, solution to the cell problem (2.16) for $\lambda = \bar{H}(\bar{x}, D_x\psi(\bar{t},\bar{x}))$.

By (2.23) and the definition (2.11) of χ, we have for η fixed

$$w(y) + \eta\chi(y) \to +\infty \text{ as } |y| \to +\infty.$$

Then, there exists $(t_{\varepsilon,\eta}, x_{\varepsilon,\eta}, y_{\varepsilon,\eta}) \in [0,T] \times \mathbb{R}^n \times \mathbb{R}^m$ point of maximum of Φ defined in (2.118). We denote

$$(t_{\varepsilon,\eta}, x_{\varepsilon,\eta}, y_{\varepsilon,\eta}) =: (t,x,y).$$

Since v_ε is a solution of equation (2.114), we test it as a subsolution with the function $\psi + \varepsilon(w + \eta\chi)$ and by writing

$$|\tau(y)^T(Dw(y) + \eta D\chi(y))|^2 = |\tau(y)^T Dw(y)|^2 + \eta^2 |D\chi(y)|^2 + 2\eta(\tau(y)^T Dw(y), D\chi(y)),$$

we get

$$\begin{aligned}\psi_t(t,x) - \varepsilon \mathrm{tr}(\sigma\sigma(x,y)^T D^2_{xx}\psi(t,x)) - \varepsilon\phi(x,y)\cdot D_x\psi(t,x) - |\sigma(x,y)^T D_x\psi(t,x)|^2 - b(y)\cdot Dw(y)\\ - \mathrm{tr}(\tau(y)\tau(y)^T D^2 w(y)) - 2\tau(y)^T\sigma(x,y)^T D_x\psi(t,x)\cdot Dw(y) - |\tau(y)^T Dw(y)|^2 + \eta G_{\varepsilon,\eta}(x,y) \leq 0,\end{aligned} \tag{2.119}$$

where, for convenience of notations, we denote

$$G_{\varepsilon,\eta}(x,y) = -b(y)\cdot D\chi(y) - \mathrm{tr}(\tau(y)\tau(y)^T D^2\chi(y)) - \eta|\tau(y)^T D\chi(y)|^2 \\ - 2\tau(y)^T Dw(y)\cdot D\chi(y) - 2\tau(y)\sigma(x,y)^T D_x\psi(t,x)\cdot D\chi(y). \quad (2.120)$$

We recall that the corrector w is solution of the cell problem (2.16) for $\lambda = \bar{H}(\bar{x}, D_x\psi(\bar{t},\bar{x}))$ (see Proposition 2.3.3), that is, w satisfies

$$\bar{H}(\bar{x}, D\psi(\bar{t},\bar{x})) = b(y)\cdot Dw(y) + \mathrm{tr}(\tau(y)\tau(y)^T D^2 w(y)) + |\tau(y)^T Dw(y)|^2 \\ + 2(\tau(y)\sigma(\bar{x},y)^T D_x\psi(\bar{t},\bar{x}))\cdot Dw(y) + |\sigma(\bar{x},y)^T D_x\psi(\bar{t},\bar{x})|^2. \quad (2.121)$$

We use (2.121) in (2.119) and we get

$$\psi_t(t,x) - \varepsilon\mathrm{tr}(\sigma\sigma(x,y)^T D^2_{xx}\psi(t,x)) - \varepsilon\phi(x,y)\cdot D_x\psi(t,x) + \eta G_{\varepsilon,\eta}(x,y) + F_\varepsilon(x,y) \\ - \bar{H}(\bar{x}, D\psi(\bar{t},\bar{x})) \le 0, \quad (2.122)$$

where we denote

$$F_\varepsilon(x,y) = (-2\tau(y)\sigma(x,y)^T D_x\psi(t,x) + 2\tau(y)\sigma(\bar{x},y)^T D_x\psi(\bar{t},\bar{x}))\cdot Dw(y) \\ - |\sigma(x,y)^T D_x\psi(t,x)|^2 + |\sigma(\bar{x},y)^T D_x\psi(\bar{t},\bar{x})|^2. \quad (2.123)$$

In the following lemma we prove that (x,t,y) are uniformly bounded in ε and that $x,t \to \bar{x},\bar{t}$ as $\varepsilon \to 0$. Note that we split the proof of the equiboundedness of (t,x,y) into (i) and (ii) in the following lemma only for convenience of exposition.

Lemma 2.7.2. *Let $\eta > 0$ be fixed. Under the above notations and under the assumptions of Theorem 2.7.1, we have*

(i) (x,t) *are uniformly bounded in* ε;
(ii) y *is uniformly bounded in* ε;
(iii) $(x,t) \to (\bar{x},\bar{t})$ *as* $\varepsilon \to 0$.

We split the proof into three steps; in Step 1 we prove (i), in Step 2 we prove (ii) and in Step 3 we prove (iii).

Proof of Lemma 2.7.2.

Step. *1 (Proof of (i))* For all $x' \in \mathbb{R}^n, y' \in \mathbb{R}^m$ and $t' \in (0,T)$ we have

$$v^\varepsilon(t,x,y) - \psi(t,x) - \varepsilon(w(y) + \eta\chi(y)) \ge v^\varepsilon(t',x',y') - \psi(t',x') - \varepsilon(w(y') + \eta\chi(y')),$$

that is

$$\psi(t,x)+\varepsilon(w(y)+\eta\chi(y)) \le 2\sup_{\varepsilon}||v^{\varepsilon}||_{\infty}+\sup_{\varepsilon}\left[\psi(t',x')+\varepsilon(w(y')+\eta\chi(y'))\right]$$

then

$$\sup_{\varepsilon}\left[\psi(t,x)+\varepsilon(w(y)+\eta\chi(y)))\right] < \infty. \tag{2.124}$$

Note that (2.124) implies

$$\sup_{\varepsilon}\psi(t,x) < \infty. \tag{2.125}$$

Indeed, (2.125) follows immediately from (2.124) if $|y|$ is bounded in ε; when $|y| \to +\infty$ it follows since $\varepsilon(w(y)+\eta\chi(y)))$ is positive thanks to the definition of χ and the logarithmic growth of w proved in (2.23). Then the uniform boundedness of x and t follows from (2.125) and the coercivity of ψ.

Step. *2 (Proof of (ii))* We proceed by contradiction, supposing $|y| \to +\infty$ as $\varepsilon \to 0$ and we get a contradiction with the equation (2.122) by applying Lemma 2.7.3, whose proof is postponed at the end of the proof of a). We just observe that it essentially relies on (i) of Lemma 2.7.2 proved in step 1, on the quadratic growth of the Lyapounov function χ and on the uniform estimate of the gradient of the corrector w (Proposition 2.5.6).

Lemma 2.7.3. *Let assumptions of Theorem 2.7.1 hold. Let $G_{\varepsilon,\eta}(x,y)$ and $F_{\varepsilon}(x,y)$ be defined respectively in* (2.120) *and* (2.123) *and let $\eta > 0$ be fixed. Then, if*

$$|y| \to +\infty \text{ as } \varepsilon \to 0, \tag{2.126}$$

then we have

(1) $\lim_{\varepsilon\to 0} G_{\varepsilon,\eta}(x,y) = +\infty.$

(2) $|\lim_{\varepsilon\to 0} F_{\varepsilon}(x,y)| \le C'$, *for some constant $C' > 0$.*

Then the uniform boundedness of y follows by coupling (1) and (2) of Lemma 2.7.3 with equation (2.122) and observing that ϕ and σ are bounded, t,x are uniformly bounded in ε and the time derivative of ψ is bounded by (2.117).

Step. *3 (Proof of (iii))* Note that, by Step 1 and Step 2, we can suppose that there exists $(\tilde{t},\tilde{x},\tilde{y})$ such that, up to subsequences

$$(t,x,y) \to (\tilde{t},\tilde{x},\tilde{y}) \quad \text{as } \varepsilon \to 0. \tag{2.127}$$

Since, for all t',x',y',

$$v^{\varepsilon}(t,x,y)-\psi(t,x)-\varepsilon(w(y)+\eta\chi(y)) \ge v^{\varepsilon}(t',x',y')-\psi(t',x')-\varepsilon(w(y')-\eta\chi(y')),$$

using the uniform boundedness of y and the definition of upper semi-limit we get

$$\bar{v}(\tilde{t},\tilde{x}) - \psi(\tilde{t},\tilde{x}) \geq \bar{v}(t',x') - \psi(t',x') \quad \forall t',x'.$$

Then

$$\tilde{x} = \bar{x}, \quad \tilde{t} = \bar{t}$$

and

$$t \to \bar{t}, \quad x \to \bar{x} \quad \text{ as } \varepsilon \to 0, \tag{2.128}$$

concluding the proof of the lemma.

□

Now we conclude the proof of Theorem 2.7.1 *a*).

Note that from now on when we do the limit as $\varepsilon \to 0$, we mean the limit along the subsequences such that (2.127) (and then also (2.128)) hold.

Note that, by (iii) of Lemma 2.7.2 and by definition of the corrector w, we have

$$\lim_{\varepsilon \to 0} F_\varepsilon(x,y) = 0, \tag{2.129}$$

where F_ε is defined in (2.123). Then, we let $\varepsilon \to 0$ in (2.122) and use again (2.128), (2.127) and (2.129) to get

$$\psi_t(\bar{t},\bar{x}) + \eta G_\eta(\bar{x},\tilde{y}) - \bar{H}(\bar{x}, D\psi(\bar{t},\bar{x})) \leq 0. \tag{2.130}$$

where

$$G_\eta(\bar{x},\tilde{y}) := \lim_{\varepsilon \to 0} G_{\varepsilon,\eta}(x,y),$$

where and $G_{\varepsilon,\eta}$ is defined in (2.120).

Note that

$$\begin{aligned} G_\eta(\bar{x},\tilde{y}) = &-b(\tilde{y}) \cdot D\chi(\tilde{y}) - \operatorname{tr}(\tau(\tilde{y})\tau(\tilde{y})^T D^2\chi(\tilde{y})) - \eta|\tau(\tilde{y})^T D\chi(\tilde{y})|^2 \\ &- 2\tau(\tilde{y})^T Dw(\tilde{y}) \cdot D\chi(\tilde{y}) - 2\tau(\tilde{y})\sigma(\bar{x},\tilde{y})^T D_x\psi(\bar{t},\bar{x}) \cdot D\chi(\tilde{y}). \end{aligned}$$

We observe that if $\tilde{y}$ is uniformly bounded in η, we send $\eta \to 0$ and we conclude

$$\psi_t - \bar{H}(\bar{x}, D\psi(\bar{t},\bar{x})) \leq 0. \tag{2.131}$$

Otherwise, if

$$|\tilde{y}| \to +\infty \text{ as } \eta \to 0,$$

we prove analogously as in Lemma 2.7.3 (1) that for any η small enough

$$\lim_{\eta\to 0} G_\eta(\bar{x},\tilde{y}) = +\infty.$$

Then we can suppose for η small

$$\eta G_\eta(\bar{x},\tilde{y}) \geq 0 \tag{2.132}$$

and by coupling (2.132) with (2.130), we conclude again (2.131).

Now we prove Lemma 2.7.3.

Proof of Lemma 2.7.3. First we prove (1). Take $\eta,\varepsilon < 1$ and consider $|y| \geq R_1$, where R_1 is defined in (U). We analyse $G_{\varepsilon,\eta}$ term by term:

$$-b(y)\cdot D_y\chi(y) \geq 2a|y|^2 - 2a|\mu||y|,$$

by assumption $1)$ of (U);

$$-|\tau(y)^T D_y\chi(y)|^2 \geq -4a^2T|y|^2$$

where $T > 0$ is defined in (2.12);

$$-2\tau(y)\sigma(x,y)^T D_x\psi(t,x)\cdot D_y\chi(y) \geq -2aK|D_x\psi(t,x)||y| - 2aK$$

where from now on we denote by $K > 0$ a constant depending only on $||\tau||_\infty, ||\sigma||_\infty$ which may change from line to line. Note that $|D_x\psi(t,x)|$ is bounded uniformly in ε by Lemma 2.7.2 (i) and the smoothness of ψ.

We control the growth of the gradient of w by the global estimate (2.69) proved in Proposition 2.5.6 and we get

$$-2\tau(y)^T D_y\chi(y)\cdot\tau(y)^T D_y w(y) \geq -4aCK|y|$$

where C is defined in (2.69).

Then, by coupling all the previous estimate we get

$$G_{\varepsilon,\eta}(x,y) \geq (2a-4a^2T)|y|^2 - 2a\mu|y| - 4aCK|y| - 2aK|D_x\psi(t,x)||y| - 2aK.$$

and by (2.13), we finally get (1).

In order to prove (2), we use again (2.69) of Proposition 2.5.6 to estimate the term

$$\tau(y)\sigma(\bar{x},y)^T D_x\psi(t,x))\cdot D_y w(y) \geq -KC|D_x\psi(t,x))|$$

where $C > 0$ is defined in (2.69). Then we conclude since (t,x) are bounded in ε by Lemma 2.7.2 (i) and τ,σ are bounded. $\square$

Remark 2.7.4. In the supercritical case $\alpha > 2$, the proof is essentially the same as above by replacing ε by $\varepsilon^{\alpha-1}$ and considering

$$\Phi = v^{\varepsilon}(t,x,y) - \psi(t,x) - \varepsilon^{\alpha-1}(w(y) + \eta\chi(y)) \tag{2.133}$$

where now w is the solution to the cell problem (2.35).

The rest of the proof follows analogously to the case $\alpha = 2$. In particular when testing v^{ε} as a subsolution with the function $\psi + \varepsilon(w + \eta\chi)$, we get

$$\psi_t(t,x) - \varepsilon \mathrm{tr}(\sigma\sigma(x,y)^T D^2_{xx}\psi(t,x)) - \varepsilon\phi(x,y)\cdot D_x\psi(t,x) + \eta G_{\varepsilon,\eta}(x,y) + F_{\varepsilon}(x,y) \\ - \bar{H}(\bar{x}, D\psi(\bar{t},\bar{x})) \leq 0,$$

where now we denote

$$G_{\varepsilon,\eta}(x,y) = -b(y)\cdot D_y\chi(y) - \mathrm{tr}(\tau(y)\tau(y)^T D^2\chi(y)) - \eta\varepsilon^{\alpha-2}|\tau(y)^T D_y\chi(y)|^2 \\ - 2\varepsilon^{\alpha-2}(\tau(y)^T D_y\chi(y), \tau(y)^T D_y w) - 2\varepsilon^{\frac{\alpha}{2}-1}(\tau(y)\sigma(x,y)^T D\psi(t,x))D_y\chi(y)$$

and

$$F_{\varepsilon}(x,y) = -\varepsilon^{\alpha-2}|\tau(y)^T D_y w(y)|^2 - 2\varepsilon^{\frac{\alpha}{2}-1}\tau(y)\sigma(x,y)^T D\psi(t,x)\cdot D_y w(y) \\ - |\sigma(x,y)^T D_x\psi(t,x)|^2 + |\sigma(\bar{x},y)^T D_x\psi(\bar{t},\bar{x})|^2$$

and we used that the corrector w satisfies the cell problem (2.35) for $\lambda = \bar{H}(\bar{x}, D\psi(\bar{t},\bar{x})$ (see Proposition 2.4.3), that is

$$\bar{H}(\bar{x}, D\psi(\bar{t},\bar{x})) = b(y)\cdot D_y wy) + \mathrm{tr}(\tau(y)\tau(y)^T D^2_{yy} w(y)) + |\sigma(\bar{x},y)^T D_x\psi(\bar{t},\bar{x})|^2.$$

Now we prove $b)$ of Theorem 2.7.1.

Step. *2 (Proof of b))* For the supersolution the argument is analogous and we just skecth the main steps for completeness. We treat at the same time the critical and supercritical case and we take $\alpha \geq 2$. We take $-\psi$ coercive in the variable x and satisfying (2.117) and $(\bar{t},\bar{x})$ a point of mimimum of $\underline{v}(t,x) - \psi(t,x)$.

Let $\eta > 0$ and take $(t_{\varepsilon,\eta}, x_{\varepsilon,\eta}, y_{\varepsilon,\eta}) \in [0,T]\times\mathbb{R}^n\times\mathbb{R}^m$ points of mimima of

$$v^{\varepsilon}(t,x,y) - \psi(t,x) - \varepsilon^{\alpha-1}(w(y) - \eta\chi(y))$$

which exists thanks to (2.23) and the definition of the Liapounov function and let

$$(t_{\varepsilon,\eta}, x_{\varepsilon,\eta}, y_{\varepsilon,\eta}) =: (t,x,y).$$

Now we test v_ε (as a supersolution) with the function $\psi+\varepsilon^{\alpha-1}(w-\eta\chi)$ and we get

$$\psi_t(t,x)-\varepsilon\mathrm{tr}(\sigma\sigma(x,y)^T D^2_{xx}\psi(t,x))-\varepsilon\phi(x,y)\cdot D_x\psi(t,x)+\eta G_{\varepsilon,\eta}(x,y)+F_\varepsilon(x,y) \\ -\bar{H}(\bar{x},D\psi(\bar{t},\bar{x}))\geq 0. \quad (2.134)$$

where

$$G_{\varepsilon,\eta}(x,y)=b(y)\cdot D\chi(y)+\mathrm{tr}(\tau(y)\tau(y)^T D^2\chi(y))-\eta|\tau(y)^T D\chi(y)|^2 \\ +2\tau(y)^T Dw(y)\cdot D\chi(y)+2\tau(y)\sigma(x,y)^T D\psi(t,x)\cdot D\chi(y), \quad (2.135)$$

and F_ε is the same as a) and for convenience we recall the definition

$$F_\varepsilon(x,y)=(-2\tau(y)\sigma(x,y)^T D\psi(t,x)+2\tau(y)\sigma(\bar{x},y)^T D\psi(\bar{t},\bar{x}))\cdot Dw(y) \\ -|\sigma(x,y)^T D_x\psi(t,x)|^2+|\sigma(\bar{x},y)^T D_x\psi(\bar{t},\bar{x})|^2. \quad (2.136)$$

The proof of Lemma 2.7.2 is analogous. In particular (2.124) becomes

$$\inf_\varepsilon\left[\psi(t,x)+\varepsilon^{\alpha-1}(w(y)-\eta\chi(y)))\right]>-\infty$$

and we conclude since $-\psi$ is coercive in x and by the fact that $\varepsilon^{\alpha-1}(w(y)-\eta\chi(y)))$ is negative (at least for when $|y|\to+\infty$).

In order to prove that y is uniformly bounded in ε, we proceed as in a) by contradiction, supposing $|y|\to+\infty$ and we get a contradiction with equation (2.134) by applying Lemma 2.7.5. We state Lemma 2.7.5 and we omit the proof, since it is analogous to that of Lemma 2.7.3.

Lemma 2.7.5. *Let assumptions of Theorem 2.7.1 hold. Let $G_{\varepsilon,\eta}$ and F_ε be defined respectively in* (2.135) *and* (2.136) *and let $\eta>0$ be fixed. If*

$$|y|\to+\infty \text{ as } \varepsilon\to 0,$$

then we have

(1) $\limsup_{\varepsilon\to 0} G_{\varepsilon,\eta}(x,y)=-\infty$.

(2) $|\limsup_{\varepsilon\to 0} F_\varepsilon(x,y)|\leq C'$, *for some constant* $C'>0$.

Then, we send $\varepsilon\to 0,\eta\to 0$ in (2.134) and we conclude similarly as in a).

Now we prove c) of Theorem 2.7.1.

Step. *3 (Proof of c))* By the definition of semilimits we have $\underline{v} \le \bar{v}$ in $[0,T) \times \mathbb{R}^n$. The comparison principle (Theorem 2.6.1) for the effective equation (2.116) gives the inequality

$$\bar{v}(t,x) \le \underline{v}(t,x),$$

then we get

$$\bar{v} = \underline{v} = v \quad \text{in } [0,T) \times \mathbb{R}^n.$$

Thanks to the properties of semilimits, we finally get that v^{ε} converges locally uniformly to the unique bounded solution of (2.116).

$\square$

Appendix A Notions of large deviation theory

We recall some standard notions of large deviation theory that we use in Chapter 1, Section 1.6.

Throughout the section, μ_ε will denote a family of probability measures defined on $\mathbb{R}^n$ with its Borel σ-field $\mathscr{B}$. For the definitions and theorems in a more general setting and for further details we refer to [71].

Given a family of probability measures $\{\mu_\varepsilon\}$, a large deviation principle characterizes the limiting behavior, as $\varepsilon \to 0$, of $\{\mu_\varepsilon\}$ in terms of a rate function through asymptotic upper and lower exponential bounds on the values that μ_ε assigns to measurable subsets of $\mathbb{R}^n$.

Definition 2.7.6. *A rate function I is a lower semicontinuous map $I : \mathbb{R}^n \to [0,\infty]$, and it is a good rate function if for all $\alpha \in [0,\infty)$, the level set $\Psi_I(\alpha) := \{x : I(x) \leq \alpha\}$ is compact.*

For any set $B \subseteq \mathbb{R}^n$, we denote by B° the interior of B.

Definition 2.7.7. *A family of probability measures $\{\mu_\varepsilon\}$ satisfies the large deviation principle with a rate function I if, for all $B \in \mathscr{B}$,*

$$- \inf_{x\in B^\circ} I(x) \leq \liminf_{\varepsilon\to 0} \varepsilon \log \mu_\varepsilon(B) \leq \limsup_{\varepsilon\to 0} \varepsilon \log \mu_\varepsilon(B) \leq - \inf_{x\in \bar{B}} I(x). \tag{2.137}$$

The right-and left-hand sides of (2.137) are referred to as the upper and lower bounds, respectively.

Definition 2.7.8. *A family of probability measures $\{\mu_\varepsilon\}$ on $\mathbb{R}^n$ is exponentially tight if for every $\alpha < \infty$, there exists a compact set $K_\alpha \subset \mathbb{R}^n$ such that*

$$\limsup_{\varepsilon\to 0} \varepsilon \log \mu_\varepsilon(K_\alpha^c) < -\alpha.$$

Moreover, for each Borel measurable function $h : \mathbb{R}^n \to \mathbb{R}$, define

$$\Lambda_h^\varepsilon := \varepsilon \log \int_{\mathbb{R}^n} e^{\frac{h(x)}{\varepsilon}} \mu_\varepsilon(dx).$$

and

$$\lim_{\varepsilon \to 0} \varepsilon \log \int_{\mathbb{R}^n} e^{\frac{h(x)}{\varepsilon}} \mu_\varepsilon(dx) = \Lambda_h \tag{2.138}$$

provided the limit exists. Then, the so-called Bryc's inverse Varadhan Lemma permits to derive the large deviation principle as a consequence of exponential tightness of the measures μ_ε and the existence of the limits (2.138) for every $h \in BC(\mathbb{R}^n)$. The statement is the following.

Lemma 2.7.9. *Suppose that the family* $\{\mu_\varepsilon\}$ *is exponentially tight and that the limit in* (2.138) *exists for every* $h \in BC(\mathbb{R}^n)$. *Then* $\{\mu_\varepsilon\}$ *satisfies the LDP with the good rate function*

$$I(x) = \sup_{h \in BC(\mathbb{R}^n)} \{h(x) - \Lambda_h\}.$$

Furthermore, for every $h \in BC(\mathbb{R}^n)$,

$$\Lambda_h = \sup_{x \in \mathbb{R}^n} \{h(x) - I(x)\}.$$

Finally we recall the optional sampling theorem. For further details see [155].

Theorem 2.7.10. *Let* $M = \{M_t\}_{t \geq 0}$ *be a submartingale right-continuous and let* τ *be a stopping time, such that one of the following conditions is satisfied*

- τ *is a.s. bounded, i.e. there exists* $T \in (0, \infty)$ *such that* $\tau \leq T$ *a.s.;*
- τ *is a.s. finite and* $M_{\tau \wedge t} \leq Y$ *for all* $t \geq 0$, *where* Y *is an integrable variable (in particular* $|M_{\tau \wedge n}| \leq K$ *for a constant* $K \in [0, \infty)$*)*

Then the variable M_τ *is integrable and*

$$E(M_\tau) \geq E(M_0). \tag{2.139}$$

If, instead, M *is a supermartingale, then*

$$E(M_\tau) \leq E(M_0).$$

Appendix B Notions of ergodic theory

We recall some basic notions of classical ergodic theory which we use in Chapter 1, Section 1.2.

Let $\mathbb{T}^m$ be the m-dimensional torus, obtained by identifying the opposite faces of $(0,1)^m$. Let $(Y(t))_{t\geq 0}$ be a stochastic process solution of

$$\begin{cases} dY_t = b(Y_t)dt + \sqrt{2}\tau(Y_t)dW_t \\ Y_0 = y \in \mathbb{T}^m, \end{cases} \tag{2.140}$$

where W_t is a standard m-dimensional Brownian motion, $b : \mathbb{R}^m \to \mathbb{R}^m$ and $\tau : \mathbb{R}^m \to \mathbf{M}^{m,m}$ are Lipschitz continuous, $\mathbb{Z}^m$ periodic with respect to the variable y.

Definition 2.7.11. *A Radon measure μ on the torus is an invariant measure for the process Y_t if*

$$\int_{\mathbb{T}^m} f(Y_t)d\mu(y) = \int_{\mathbb{T}^m} f(y)d\mu(y), \quad \text{for every } f \in C(\mathbb{T}^m), t > 0.$$

Definition 2.7.12. *The process Y_t is said to be ergodic if, given an invariant measure μ, we have for every $f \in L^1(\mu)$*

$$E\frac{1}{T}\int_0^T f(Y_t)dt \to \int_{\mathbb{T}^m} f(y)d\mu(y) \text{ as } T \to +\infty \text{ for } \mu\text{-almost every } y.$$

The process Y_t is said to be uniquely ergodic is there exists a unique invariant probability measure.

Remark 2.7.13. Suppose τ uniformly non-degenerate in (2.140). Then, Y_t is uniquely ergodic (we refer to [3] for a proof).

The next proposition gives another characteriation of the ergodicity of the process Y_t. For the proof we refer to [2].

Proposition 2.7.14. *There exists a unique invariant probability Radon measure for the process Y_t if and only if, for every $f \in C(\mathbb{T}^m)$,*

$$E\frac{1}{T}\int_0^T f(Y_t)dt \to const \quad \text{as } T \to +\infty, \text{ uniformly in } y$$

Moreover, the constant is $\int_{\mathbb{T}^m} f(y)d\mu(y)$.

We give the following characterization of the invariant measure of Y_t in terms of the periodic solution of a partial differential equation. For the proof we refer to Bensoussan, J.-L. Lions, Papanicolaou [38], Jensen, Lions [114] and Evans [76].

Lemma 2.7.15. *Let Y_t have values on the torus $\mathbb{T}^m$ and let μ be the invariant measure associated with Y_t. Then μ is the periodic solution of*

$$-\sum_{i,j} \frac{\partial^2}{\partial y_i \partial y_j}((\tau\tau^T)_{ij}(y))\mu + \sum_i \frac{\partial}{\partial y_i}(b_i(y))\mu = 0 \quad \text{in } \mathbb{R}^m, \tag{2.141}$$

with mean $\int_{\mathbb{T}^m} \mu(y)dy = 1$.

Remark 2.7.16. We observe that form classical results based on the Fredholm alternative (see, for instance, [38] or [37]), for b and τ smooth in y, there is a unique solution μ of (2.141) with average 1.

A general Abelian-Tauberian theorem

For $(\bar{x},\bar{p},\bar{X}) \in (\mathbb{R}^n,\mathbb{R}^n,\mathbf{M}^{n,n})$ fixed, let $F(\bar{x},\bar{p},\bar{X})$ be a continuous Hamiltonian on $\mathbb{R}^m \times \mathbb{R}^m \times \mathbf{M}^{m,m}$. Consider a general cell problem of finding the unique constant $\bar{H}(\bar{x},\bar{p},\bar{X})$ such that there exists a viscosity solution w of the following equation

$$\bar{H}(\bar{x},\bar{p},\bar{X}) + F(\bar{x},y,\bar{p},Dw,\bar{X},D^2w) = 0 \quad \text{in } \mathbb{R}^m, w \text{ periodic}. \tag{2.142}$$

For $\delta > 0$, consider w_δ solution of the approximate stationary cell problem

$$\delta w_\delta + F(y,\bar{x},\bar{p},Dw_\delta,\bar{X},D^2w_\delta) = 0 \quad \text{in } \mathbb{R}^m, \tag{2.143}$$

and the Cauchy problem

$$w_t + F(y,\bar{x},\bar{p},Dw,\bar{X},D^2w) = 0, \text{ in } (0,+\infty)\times\mathbb{R}^m, w(0,\cdot) = 0 \text{ in } \mathbb{R}^m. \tag{2.144}$$

The following theorem states that the solvability of the cell problem (2.142) is equivalent to convergence of δw_δ and $\frac{w(t,\cdot)}{t}$ to a constant respectively as $\delta \to 0$ and $t \to +\infty$.

The equivalence $(i) \Leftrightarrow (ii)$ can be viewed as a generalized Abelian-Tauberian theorem (see [140]). It was proved in [8] (see also [14]) for first-order HJB equations and extended in [9] to second-order HJB equations; these papers exploited the optimal control interpretations of the solutions and used the dynamic programming principle. We refer to [2] for a proof valid for an arbitrary Hamiltonian which only uses the comparison principle and the theory of viscosity solutions.

Theorem 2.7.17. *The following statements are equivalent:*

(i) If w_δ is the solution of the stationary problem (2.143)*, then*

$$\delta w_\delta \to \ const \quad uniformly\ in\ y\ as\ \delta \to 0;$$

(ii) if w is the solution of the Cauchy problem (2.144)*, then*

$$\frac{w(t,\cdot)}{t} \to \ const \quad uniformly\ in\ y\ as\ t \to +\infty;$$

(iii) there exists a unique constant $\bar{H}(\bar{x},\bar{p},\bar{X})$ such that the true cell problem (2.142) *has a periodic viscosity solution w.*

If one of the above assertion is true, then the constants in (i) and (ii) are equal and they coincide with $\bar{H}(\bar{x},\bar{p},\bar{X})$.

Part II

On Neumann problems for nonlocal Hamilton-Jacobi equations

Specific notation of Part II

Chapter 3

Some preliminary results

3.1 Introduction

In this chapter we recall some of the main results of [24], where Neumann boundary value problems are studied for linear partial-integro differential related to Lévy type processes. Before entering into the details of the results of [24], we give a brief introduction of the setting and the kind of results.

In the classical probabilistic approach to elliptic and parabolic partial differential equations, Neumann type boundary value problems are associated to stochastic processes being reflected on the boundary. Roughly speaking, a key result is that for a PDE with Neumann or oblique boundary conditions, there is a unique underlying reflection process and any consistent approximation will converge to it in the limit (see [130] and Barles, Lions [30]). At least in the case of normal reflections, this result is strongly connected to the study of the Skorohod problem and relies on the underlying stochastic processes being continuous.

In the setting of [24], the underlying processes are of Lévy type, that is, they are discontinuous and can jump. Here the situation is different and must be addressed in a different way. Indeed, for jump processes which may exit the domain without first having hit the boundary, there are many ways to define a “reflection”or a “reflecting process”.

The paper [24] studies linear equations of the type

$$\begin{cases} u(x) - \mathscr{I}[u](x) + f(x) = 0 & \text{in } \mathscr{H} \\ -\frac{\partial u}{\partial x_n} = 0 & \text{on } \partial\mathscr{H}. \end{cases} \tag{3.1}$$

where by $\mathscr{H}$ we denote the halfspace, that is,

$$\mathscr{H} = \{x = (x_1, \cdots, x_n) = (x', x_n) \in \mathbb{R}^n, x_n > 0\},$$

f is a bounded continuous function and

$$\mathscr{I}[u](x) = \lim_{\delta \to 0^+} \int_{|z| \geq \delta} u(x + j(x,z)) - u(x) d\mu(z), \tag{3.2}$$

where μ is a singular nonnegative Radon measure satisfying

$$\int_{\mathbb{R}^n} (1 \wedge |z|^2) d\mu(z) < +\infty \tag{3.3}$$

and

$$x + j(x,z) \in \bar{\mathscr{H}} \text{ for all } x \in \bar{\mathscr{H}}, \quad j(x,z) = z \text{ if } x+z \in \bar{\mathscr{H}}. \tag{3.4}$$

Note that $\mathscr{I}[u]$ is a principal value (P.V.) integral and, when $j(x,z) \equiv z$, then $\mathscr{I}[u]$ is the generator of a stochastic process which can jump from $x \in \bar{\mathscr{H}}$ to $x+z$ with a certain intensity, see for example [6], [62], [92]. Condition (3.3) is the most general integrability assumption satisfied by Lévy measures (see [6]). Assumption (3.4) is a type of reflection condition preventing the underlying process from leaving the domain: nothing happens and $j(x,z) = z$ if $x+z \in \bar{\mathscr{H}}$, while if $x+z \notin \bar{\mathscr{H}}$, then a "reflection" is performed in order to move the particle back to a point $x + j(x,z)$ inside. Note that, because of the way the PIDE and the process are related, defining a reflection on the boundary influences the nature of the nonlocal term (3.2) and then changes the equation inside the domain. This is a new nonlocal phenomenon which is not encountered in the case of continuous processes and PDEs.

In [24] different models of reflection are presented, we refer to the following section where we describe them in details.

Finally we remark that equation (3.1) is interpreted in the sense of generalized viscosity solution which provides a suitable definition of "generalized " Neumann boundary condition. This means that in certain cases the equation could hold up to the boundary and the Neumann condition could not be attained, in other words the underlying process could not reach the boundary. For the precise definition of solutions of the problem (3.1) we refer to Definition 2.3 of [24].

3.2 Four models of reflection

In [24] four principal models of reflection are considered, namely the *mirror projection*, *normal projection*, *fleas on the window* and *censored* model.

Essentially, we proceed this way: if the process jumps from x to $x+z$ exiting the domain, then $x+z$ is replaced by a point $x + j(x,z)$ situated inside the domain or on the boundary. The difference between the four models is the way the process is forced to rest inside, represented by the function j. In figure (3.1) we represent the four models of reflection and we denote by $j_i, i = 1, \cdots 4$ each of the reflections as explained in the following:

(1) *Mirror projection*: a reflection on the boundary is performed, i.e. we go to $x + j_1(x,z)$
(2) *Normal projection*: the process is dragged on the boundary killing the normal component, following the method of Lions-Sznitman, i.e. we go in $x + j_2(x,z)$.
(3) *Fleas on the window*: the process is stopped on the boundary as if it was "glued" on the wall (from this the name given to this model "fleas on the window"), i.e. we go to $x + j_3(x,z)$.
(4) *Censored*: in case of a jump outside the domain, the process is stopped (actually there are not jumps), then it is resuscitated from the point where it was before the jump, i.e. $x + j_4(x,z) = x$.

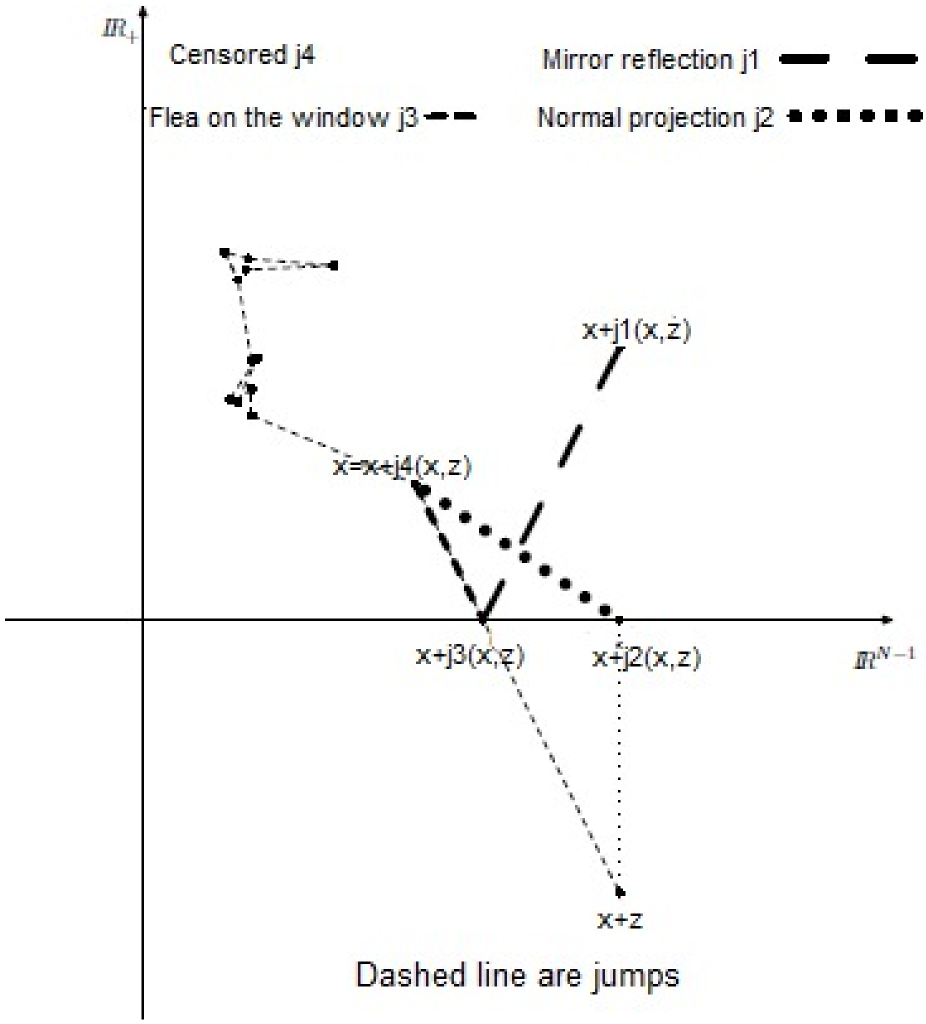

Fig. 3.1 Models of reflection

Let us denote with $x=(x',x_n)\in\mathbb{R}^{n-1}\times\mathbb{R}^+$, $j_i(x,z)=(j_i(x,z)',j_i(x,z)_n), i=1\cdots4$. More precisely, we have for all $x\in\mathscr{H}, z\neq 0$

$$
\begin{aligned}
(1)\ j_1(x,z) &= \begin{cases} z & \text{if} x_n+z_n\geq 0 \\ (z',-2x_n-z_n) & \text{if not} \end{cases}\\
(2)\ j_2(x,z) &= \begin{cases} z & \text{if} x_n+z_n\geq 0 \\ (z',-x_n) & \text{if not} \end{cases}\\
(3)\ j_3(x,z) &= \begin{cases} z & \text{if} x_n+z_n\geq 0 \\ (z',-z\frac{x_n}{|z_n|}) & \text{if not} \end{cases}\\
(4)\ j_4(x,z) &= \begin{cases} z & \text{if} x_n+z_n\geq 0 \\ 0 & \text{if not,} \end{cases}
\end{aligned}
$$

We observe that the cases 1-3 share the following properties

(i) $|j(x,z)|\leq C|z|$;

(ii) $j(x,\sigma_i z)=-j(x,z)_i$ for $i=1,\cdots n-1$ where $\sigma_i(z)=(z_1,\cdots,-z_i,\cdot,z_n)$;

(iii) $j(y,z)\to j(x,z)$ when $y\to x$ for almost all z;

(iv) $|j(x',s,z)'-j(y',s,z)'|\leq C|x'-y'||z|$ for all x',y',z and $s>0$;

(v) $|j(x,z)_n-j(y,z)_n|\leq|x_n-y_n|$.

The comparison principle for the problem (3.1) in the cases 1-3 is proved in Theorem 4.1 of [24]. We remark that in the proof a key role is played by the property (v), i.e. the Lipschitz continuity in the variable x_n.

The case (4) is far more singular and needs to be treated with a different method, essentially because $j(x,z)=0$ if $x_n+z_n<0$.

Notice that the models 1 and 3 are quite natural ways to define “ reflections” (in particular mirror reflection), but they could be problematic to work with in general domains due to the possibility of multiple reflections. We also recall that model 2 has been throughly investigated in the paper [26] for fully non-linear equations set in general domains.

In the following sections we give some more details on the results of [24] for the censored case.

3.3 The censored case

There are two main cases. The first case is when the singularity of the measure μ is not too strong, typically μ has density

$$\frac{d\mu(z)}{dz}\sim\frac{dz}{|z|^{n+\sigma}},\quad \sigma<1. \tag{3.5}$$

The second case is when the singularity is stronger, i.e. μ has density

$$\frac{d\mu(z)}{dz} \sim \frac{dz}{|z|^{n+\sigma}}, \quad \sigma \in [1,2). \tag{3.6}$$

The degree of singularity influences the nature of the boundary value problem (3.1), in the sense that the Neumann boundary condition is attained only if the measure is singular enough, say as in (3.6), corresponding to the fact that the underlying process does not reach the boundary. In this case the Neumann boundary condition intervenes to treat the case of maximum point situated on the boundary. On the contrary, if the singularity is weak, say as in (3.5), the Neumann boundary condition is simply encoded in the operator and the equation holds also at the boundary. In particular in this case it is proved in Theorem $A.2$ of [24] that there is a "blow-up" solution, which implies that in the proof of the comparison principle there is no need of caring about the points on the boundary.

For precise definitions of viscosity solutions in the generalized sense and for a comparison between the case of linear equations as (3.1) and the case of nonlinear equations as considered in Chapter 4, we refer respectively to Definition 2.3 of [24] and to Definition 4.2.5, Chapter 4.

Not too singular measures

In the case of a weak singularity, the comparison principle is proved in Theorem 5.2 of [24]. The proof of the theorem relies on the existence of a solution exploding on the boundary, as described in the following assumption.

(U) There exists $R_0 > 0$ such that, for any $R > R_0$, there is a positive function $U_R \in C^2(\Omega)$ such that

$$-\mathscr{I}[U_R](x) \geq -K_R \quad \text{in } \{x : 0 \leq x_n \leq R\}$$

for some $K_R \geq 0$ and

$$U_R(x) \geq \frac{1}{\omega_R(x_n)} \quad \text{in } \Omega$$

for some function ω_R which is nonnegative, continuous, stricly increasing in a neighbourhood of 0 and satisfies $\omega_R(0) = 0$.

Tthe proof of Theorem 5.2 of [24] relies deeply on the use of the function U of assumption (U) in the penalization, that is, in the study of

$$\sup_{x,y} \left(u(x) - v(y) - \frac{|x-y|^2}{\varepsilon^2} - kU(x) - kU(y) + \text{ localization terms} \right)$$

Notice that these "blow-up " terms, namely $-kU(x)-kU(y)$, avoid that the points of maximum to be on the boundary. Then the proof is quite standard since we can use the equation inside the domain to obtain the comparison.

The existence of a function as in (U) is proved for a wide range of measures among which the case of the fractional laplacian. We refer in particular to Lemma $A.1$ and Theorem $A.2$ of [24]. We remark that when $d\mu(z)=\frac{dz}{|z|^{n+\alpha}}$ for $\sigma\in(0,1)$ (see Lemma $A.1$ of [24]), it is enough to take

$$\mathscr{U}(x)=-\ln(x_n)$$

and in this case

$$-\mathscr{I}[\mathscr{U}](x)>0 \quad \text{for } x\in\Omega.$$

For more general measures we refer to Theorem $A.2$ of [24]. An example is when

$$\frac{d\mu}{dz}=\frac{g(z)}{|z|^{n+\sigma}} \text{ where } \begin{cases} \sigma\in(0,1), \\ 0\le g\in L^\infty(\mathbb{R}^n) \\ \lim_{z\to 0} g(z)=g(0)>0. \end{cases}$$

Note that L^∞ assumption makes $\frac{d\mu}{dz}$ integrable near infinity.

Remark 3.3.1. We remark that analogously to Theorem $A.2$ of [24], we can obtain the existence of the blow-up function U also in the case of measures depending on x of the following type

$$\frac{d\mu}{dz}=\frac{g(x,z)}{|z|^{n+\sigma}}, \quad \sigma\in(0,1), \quad 0\le g\in L^\infty(\mathbb{R}^n\times\mathbb{R}^n),$$

and there exists $x_0\in\bar{\Omega}, C>0, C'>0$

$$\begin{cases} g(x_0,0)>0, \\ |g(x_0,z)-g(x_0,0)|\le C|z| \quad \forall z\in\mathbb{R}^n, \\ |g(x,z)+g(x_0,z)|\le C'|z||x-x_0| \quad \forall x\in\bar{\Omega}, z\in\mathbb{R}^n. \end{cases}$$

More singular measures

In the case of a stronger singularity, i.e. the measure μ is assumed to be the sum of two nonnegative Radon measures μ_- and μ_*

$$\mu=\mu_-+\mu_*$$

where

$$\mu_-(dz) = \frac{dz}{|z|^{n+\sigma}}, \quad \int (1 \wedge |z|^{\bar\beta})\mu_*(dz) < \infty, \quad \int_{z_n=a} d\mu_*(z) = 0 \quad \text{for all } a < 0,$$

where $\sigma \in [1,2)$ and $\bar\beta = \sigma - 1$. Note that in this case the comparison princple is less general than those in the other cases, not only for the restriction on μ_*, but also for an hypothesis of regularity *a priori* which they needed to conclude the comparison.

The result is proved in Theorem 6.1 of [24] and is the following: if u is a sub-solution and v is a super-solution such that for a certain $\beta > \bar\beta$,

$$u(x',x_n) \geq u(x',0) - C|x_n|^\beta, \quad v(x',x_n) \geq v(x',0) - C|x_n|^\beta$$

then we have $u \leq v$. In particular, their result states that there exists at most one solution of class $C^{0,\beta}$ for all β. However, they proved in dimension $n = 1$ that each bounded solution which is uniformly continuous is $C^{0,\beta}$ for a certain $\beta > \bar\beta$ and they also build such solutions (see Theorem 6.3 of [24]). In bigger dimensions, the same result holds with some additional hypothesis of regularity on f with respect to the other variables in $\mathbb{R}^{n-1}$ which are quite strong and not completely satisfying (see Theorem 6.4 of [24]).

We observe that, though the result could not be optimal, it is consistent with the "natural "Neumann boundary condition for the reflected σ-stable process (proved by Guann and Ma [101] through the variational formulation and Green type formulas) which in the case of the halfspace reads

$$\lim_{t\to 0} t^{2-\sigma} \frac{\partial u}{\partial x_n}(x + te_N) = 0. \tag{3.7}$$

This allows the normal derivative to growth less than $|x_n|^{\sigma-2}$ and then suggests that it is appropriate to look for solutions which are β-Hölder continuous, with $\beta > \sigma - 1$, as assumed in [24]. We remark that the previous argument suggests also that, on the contrary, in the case $\sigma < 1$ there is no need to assume any further regularity.

Existence and consistency of the models

In Corollary 4.2 of [24] existence of a solution for each models is obtained through standard Perron's method for nonlocal equations.

Morever, the four models are consistent with the local Neumann problem: all the proposed nonlocal models approach the "local" case $\sigma = 2$. More precisely, consider Lévy measures μ_σ with densities

$$\frac{d\mu_\sigma}{dz} = (2-\sigma)\frac{g(z)dz}{|z|^{n+\sigma}},$$

where g is a nonnegative bounded function which is C^1 in a neighbourhood of 0 and $g(0) > 0$. In this case for each nonlocal Neumann model, the solutions u_σ associated to the above sequence of measures converge as $\sigma \to 2$ to the unique viscosity solution of the same limit problem, namely

$$\begin{cases} -a\triangle u - b\cdot Du + u = f, & \text{in } \mathscr{H} \\ -\frac{\partial u}{\partial x_n} = 0 & \text{in } \partial\mathscr{H} \end{cases}$$

where $a := g(0)\frac{|S^{n-1}|}{n}$ and $b := Dg(0)\frac{|S^{n-1}|}{n}$. This proves that each model is reasonable, in the way that it is a consistent nonlocal approximation of the classical local model. This is proved in Theorem 7.1 of [24] to which we refer for details and proof.

For problems without boundary conditions, such asymptotic results have been known for a long time, we refer for example to [35], [51] and references therein for more details.

Chapter 4

Existence and uniqueness for the elliptic problem

4.1 Introduction

In the probabilistic approach to PDEs, Neumann boundary conditions are associated to stochastic processes being reflected on the boundary. The underlying idea to force the stochastic process to remain inside the domain of the equation. Classically, this is obtained essentially by a reflection on the boundary (see the method developed by Lions and Sznitman [130] in the continuous setting). A key result in the classical setting is that, for a PDE with Neumann boundary conditions, there is a unique underlying reflection process and any consistent approximation will converge to it (see [130] and Barles, Lions [30]).

When dealing with discontinuous jumping processes, the underlying idea is the same but the situation is different. This is essentially due to the fact that the jump processes may exit the domain without having first hit the boundary. The consequence is that Neumann boundary conditions can be obtained in many ways, depending on the kind of reflection we impose on the outside jumps. Moreover, the choice of a reflection on the boundary changes the equation inside the domain.

In [24] different models of reflection are presented in the framework of Neumann boundary value problems for simple linear PIDEs in domains with flat boundary (namely, a halfspace) of the type

$$\begin{cases} u(x) - \mathscr{I}[u](x) + f(x) = 0 & \text{in } \mathscr{H} \\ -\frac{\partial u}{\partial x_n} = 0 & \text{on } \partial\mathscr{H}. \end{cases} \tag{4.1}$$

where $\mathscr{H}$ is the halfspace, i.e. $\mathscr{H} = \{x = (x_1, \cdots, x_n) \in \mathbb{R}^n, x_n > 0\}$, f is a bounded continuous function and $\mathscr{I}$ represents the nonlocal diffusion and is related to the kind of reflection imposed on the boundary. We refer to Chapter 3 for more details on the results of [24]. Among the different models they consider, two types of reflections are particularly relevant for possible extensions in a more general setting. The first is the *normal projection*, close to the approach of Lions-Sznitman in [130], where outside jumps are immediately projected to the boundary by killing their normal component. This model has been thoroughly investigated in the paper [26] for fully non-linear equations set in general domains.

The second, the *censored* model, is the one we consider. In this case, any outside jump of the underlying process is cancelled (censored) and the process is restarted (resurrected) at the origin of that jump. The fact that the process is not allowed to jump outside $\bar{\Omega}$ is encoded in the definition of the nonlocal diffusion as follows

$$\mathscr{I}[u](x) = \lim_{\delta\to 0^+} \int_{\substack{|z|>\delta,\\ x+z\in\bar{\Omega}}} [u(x+z)-u(x)]d\mu(z), \tag{4.2}$$

where μ is a singular nonnegative Radon measure representing the intensity of the jumps from x to $x+z$ and satisfying some integrability condition and $\mathscr{I}$ has to be interpreted as a principal value (P.V.) integral. Note that the domain of integration is restricted to the z such that $x+z\in\bar{\Omega}$, avoiding thus any outside jump.

Note that we follow the PIDE analytical approach developed in [24], in the sense that we directly work with the infinitesimal generator and not yet with the processes themselves. For more details and probabilistic references on censored processes, we refer to e.g. [41], [90], [113], [101] and to the introduction of [24]. We just mention that the underlying processes in this paper are related to the censored stable processes of Bogdan [41] and the reflected σ-stable process of Guan and Ma [101].

We stress that the boundary value problem (4.1) is interpreted in the sense of viscosity solutions, which provides a suitable definition of "generalized" Neumann boundary condition, in the sense that in certain cases the equation could hold up to the boundary and the Neumann condition could not be attained, and this corresponds to the fact that the underlying process could not reach the boundary. We refer to Section 4.2 for the precise definitions of solutions.

In [24] it is shown, in the case of linear PIDEs, that the kind of singularity of μ influences the nature of the boundary value problem (4.1) in the sense that the Neumann boundary condition is attained only if the measure is singular enough. In particular, when the singularity is of order stricly less than 1, e.g. when

$$d\mu(z) \sim \frac{dz}{|z|^{N+\sigma}}, \quad \sigma\in(0,1), \tag{4.3}$$

the equation holds up to the boundary and process never reach the boundary.
On the other hand, when the singularity of the measure is strong, i.e. when μ is of the type (4.3) with $\sigma\in[1,2)$, the situation is far more complicated, mainly due to the "ugly"dependence in x of the operator in (4.2) and to the interplay between the singularity of

the measure and the geometry of the boundary (see Chapter 3 for more details on the case of strong singularity).

In the this Chapter we present the results of [100], where we study the well-posedness of censored type Neumann problems in the case of measures of singularity strictly less than 1, namely with $\sigma \in (0,1)$ in (4.3), in the presence of an Hamiltonian term, which forces the process to hit the boundary (and then Neumann boundary condition be attained). To be more specific, we consider the following

$$\begin{cases} u(x) - \mathscr{I}[u](x) + H(x,Du) = 0 & \text{in } \Omega \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.4}$$

where $H : \bar{\Omega} \times \mathbb{R}^N \mapsto \mathbb{R}$ is a continuous function, $\Omega \subset \mathbb{R}^N$ is an open (smooth enough) domain and $\mathscr{I}[u]$ is an *integro-differential operator of censored type and of order stricly less than* 1 defined as

$$\mathscr{I}[u](x) = P.V. \int_{x+j(x,z)\in\bar{\Omega}} [u(x+j(x,z)) - u(x)]d\mu_x(z) \tag{4.5}$$

where $d\mu_x$ is mainly of the type $g(x,z)|z|^{-(N+\sigma)}dz$ for a bounded and Lipschitz function g and $j(x,z)$ is *jump functions* $j(x,z)$, see assumptions (M0),(M1), (J0), (J1), (J2) in the following section for details.

Note that, since censored type processes are not allowed to jump outiside $\bar{\Omega}$, we don't need any condititions on Ω^c in the boundary value problem (4.4). We remark also that in the case of nonlinear equation as we consider (4.4), the situation is more complicated than in [24] even in the case of less singular measures as in (4.3) (namely with density of the type $\frac{d\mu}{dz} \sim \frac{1}{|z|^{N+\sigma}}$ with $\sigma \in (0,1)$), since the Hamiltonian term could push the process to hit the boundary and then the Neumann condition to be attained.

We consider a class of Hamiltonians with a gradient growth stronger than the diffusive term in the nonlocal operator. The first example is Hamiltonian H with *superfractional coercive* growth in the gradient variable, namely

$$H(x,p) = a(x)|p|^m - f(x), \tag{4.6}$$

where $m > \sigma$, $a, f : \bar{\Omega} \mapsto \mathbb{R}$ are bounded and continuous functions and $a(x) \geq a_0 > 0$ for some fixed constant a_0. We remark that the positivity of a and the condition $m > \sigma$ make the first-order term the leading term in the equation. We also observe that we have no other additional restriction to m (in particular, we can deal with Hamiltonians as in (4.6) with $m < 1$), allowing the study of Hamiltonians which are concave in Du.

The second main example is Hamiltonian H of *Bellman type*, which arises in the study of Hamilton-Jacobi equations associated to optimal exit time problems, such as

$$H(x,p) = \sup_{\alpha\in\mathscr{A}} \{-b(x,\alpha)\cdot p - l(x,\alpha)\}, \tag{4.7}$$

where $\mathscr{A}$ is a compact metric space (the control space) and b,l are continuous and bounded functions (we refer the reader to [14] and [85] for some connections between this type of equations and control problems). Note that the diffusive term of $\mathscr{I}$ defined in (4.5) is of weaker order than the first-order term when we assume $\sigma < 1$. We also observe that, as in [34] and [152], the well-posedness of (4.4) with Hamiltonians as in (4.7) is based on a careful study of the effects of the drift b at each point of $\partial\Omega\times(0,+\infty)$.

The main result of this Chapter is the comparison princile between bounded sub and superviscosity solutions to (4.4), see Theorem 4.2.6. We remark that the proof of this result is not standard even in the case $\sigma < 1$ in the halfspace. The difficulties are mainly due to the fact that operators as in (4.5) behave badly in x. The main idea which is behind the proof is to localize the argument on points which have the same distance from the boundary and this is carried out through the use of a non-standard non regular test functions. After the localization procedure, the rest of the proof in the case of the halfspace is simple, whereas in the case of general domains, a lot of technical difficulties arise form the way the x-depending set of integration of $\mathscr{I}$ interferes with the geometry of the boundary. To face these extra technical difficulties, we rectify the boundary relying on the smoothness of Ω. This is done in Lemma 4.3.2 which is a key result used in the proof of Theorem 4.2.6 and which we prove before Theorem 4.2.6 in Section 4.3.

The first main application of our result is the proof of existence and uniqueness for (4.4), by standard Perron's method (Corollary 4.2.7).

4.2 Assumptions, definition of solutions and main results

Assumptions

We consider $\Omega\subset\mathbb{R}^n$ such that

$$\Omega \text{ is of class } W^{2,\infty}. \tag{O}$$

This means that for any $\hat{s}\in\partial\Omega$ there exists $r = r(\hat{s})$ and a $W^{2,\infty}$-diffeomorphism

$$\psi : B(\hat{s},r)\mapsto\mathbb{R}^n, \tag{4.8}$$

satisfying

$$\psi_n(s) = d(s) \text{ for any } s\in B_r(\hat{s}), \tag{4.9}$$

where d is the signed distance from the boundary of Ω.

Remark 4.2.1. By assumption (O), there exists a neighbourhood of the boundary of Ω where the distance from the boundary d is smooth. Unless otherwise specified, in the rest of the chapter we denote by d a function which coincides with the signed distance from the boundary of $\partial\Omega$ in this neighbourhood and is bounded in all the domain. We denote by $n(x)$ the exterior unit normal vector to $\partial\Omega$ and we write $n(x) = -Dd(x)$ in the neighbourhood of the boundary where d is smooth.

We consider nonnegative Radon measures with density $\frac{d\mu_x}{dz}$ satisfying

(M0) there exists $C_\mu > 0, \sigma \in (0,1)$ such that

$$\frac{d\mu_x}{dz} \leq C_\mu |z|^{-(n+\sigma)} \quad \text{for any } x \in \bar{\Omega}, z \in \mathbb{R}^n;$$

(M1) if σ is as in (M0), there exists $D_\mu > 0$ such that for any $x, y \in \bar{\Omega}, z \in \mathbb{R}^n$

$$|\frac{d\mu_x}{dz} - \frac{d\mu_y}{dz}| \leq D_\mu |x-y||z|^{-(n+\sigma)}.$$

For example, (M0) and (M1) are satisfied for

$$d\mu_x = g(x,z)|z|^{-(n+\sigma)} dz \quad x \in \bar{\Omega}, z \in \mathbb{R}^n, \tag{4.10}$$

where $\sigma \in (0,1)$, $g : \mathbb{R}^n \times \mathbb{R}^n \mapsto \mathbb{R}$ is a nonnegative bounded function such that $g(\cdot, z)$ is Lipschitz uniformly with respect to z.

Concerning the jump function j we assume

(J0) For any $x \in \bar{\Omega}$

$$j(x, \cdot) \in C^1(\mathbb{R}^n)$$

and j is invertible such that

$$j^{-1}(x, \cdot) \in C^1(\mathbb{R}^n), \quad |Dj^{-1}(x, \cdot)| \leq A_j \quad \forall x \in \bar{\Omega}, z \in \mathbb{R}^n;$$

(J1) there exists $\tilde{C}_j, C_j > 0$ such that for any $x \in \bar{\Omega}, z \in \mathbb{R}^n$, it holds

$$\tilde{C}_j |z| \leq |j(x,z)| \leq C_j |z|;$$

(J2) there exists $D_j > 0$ such that for any $x, y \in \bar{\Omega}, z \in \mathbb{R}^n$

$$|j(x,z) - j(y,z)| \leq D_j |z||x-y|.$$

For example (J0), (J1), (J2) are satisfied for

$$j(x,z) = f(x)z \quad x \in \bar{\Omega}, z \in \mathbb{R}^n,$$

where $f : \mathbb{R}^n \mapsto \mathbb{R}$ is Lipschitz, bounded and invertible with bounded inverse.

Bellman Hamiltonian

Let $\mathscr{A}$ be a compact metric space, $b : \bar{\Omega} \times \mathscr{A} \to \mathbb{R}^n$ and $f : \bar{\Omega} \times \mathscr{A} \to \mathbb{R}$ be continuous and bounded functions. We say that H is of *Bellman type* if for $x \in \bar{\Omega}, p \in \mathbb{R}^n, H(x,p)$ can be written as

$$H(x,p) = \sup_{\alpha \in \mathscr{A}} \{-b(x,\alpha) \cdot p - l(x,\alpha)\}; \tag{4.11}$$

and satisfies the assumptions below. We assume also:

(C) *Uniform continuity of the cost l*:
There exists a modulus of continuity ω_l such that

$$|l(x,\alpha) - l(y,\alpha)| \leq \omega_l(|x-y|) \quad \forall \alpha \in \mathscr{A}, \forall x, y \in \bar{\Omega};$$

(L) *Uniform Lipschitz continuity of the drift b*:

$$(\exists C > 0)\,(\forall \alpha \in \mathscr{A})\,(\forall x, y \in \bar{\Omega}) \ : \ |b(x,\alpha) - b(y,\alpha)| \leq C|x-y|.$$

We introduce the following notations

$$\Gamma_{\text{in}} := \{x \in \partial\Omega : b(x,\alpha) \cdot n(x) < 0 \, \forall \alpha \in \mathscr{A}\}, \tag{4.12}$$

$$\Gamma_{\text{out}} := \{x \in \partial\Omega \,|\, b(x,\alpha) \cdot n(x) > 0 \quad \forall \alpha \in \mathscr{A}\}, \tag{4.13}$$

$$\Gamma := \{x \in \partial\Omega \,|\, \exists \alpha_1, \alpha_2 \in \mathscr{A} \text{ s. t. } b(x,\alpha_1) \cdot n(x) < 0, b(x,\alpha_2) \cdot n(x) > 0\}. \tag{4.14}$$

Roughly speaking, Γ_{in} and Γ_{out} can be respectively understood as the set of points where the drift term pushes inside and outside Ω the trajectories.

In order to avoid two completely different drift's behavior for arbitrarily closed points, we assume that each of these subsets is uniformly away from the others, as encoded in the following assumptions (B1) and (B2). For example, if $\partial\Omega$ is connected, then it consists in one piece belonging to one of Γ_{in}, Γ_{out} and Γ; otherwise, we are able to deal with boundary with several components of different types, precisely each one belonging to one between Γ_{in}, Γ_{out} and Γ.

The assumption we do on these subsets are the following

$$\Gamma_{\text{in}} \cup \Gamma_{\text{out}} \cup \Gamma = \partial\Omega \tag{B1}$$

and

$$\Gamma_{\text{in}}, \Gamma_{\text{out}}, \Gamma \text{ are unions of connected components of } \partial\Omega. \tag{B2}$$

Remark 4.2.2. Note that the strict sign in the definition of $\Gamma_{\text{in}}, \Gamma_{\text{out}}$ and Γ is fundamental, since it makes the Hamiltonian the leading order term in the equation, allowing us to control the growth of the nonlocal term which is of order strictly less than 1.

Remark 4.2.3. In order to treat the points of Γ_{in}, we use the existence of a blow-up supersolution exploding on the boundary. We follow the same approach of [24], where the existence of a blow-up supersolution is proved for censored type operators (of order stricly less than 1) when the measure of integration satisfies specific assumptions (in particular does not depend on x and there exists at least one point where it is strictly positive). In this particular case it is shown in [24] that the integral term computed on the blow-up supersolution does not explode on the boundary. This is not true anymore when considering more general measures as we consider in (M0), (M1). In order to solve this difficulty, we assume the strict sign in the behaviour of the drift term on Γ_{in}, which allows us to control the growth on the boundary of the integral term computed on this blow-up supersolution. We refer to the proof of Lemma 4.4.1 and in particular to Lemma 4.4.7 for further details.

Coercive Hamiltonian and Examples

We consider *superfractional* coercive Hamiltonians:

(H1) There exists $m > \sigma, c_0 > 0, D > 0$ such that for all $x \in \bar{\Omega}, p \in \mathbb{R}^n$

$$H(x,p) \geq c_0|p|^m - D.$$

We distinguish the case of sub or superlinear coercivity:

Sublinear coercivity: We say that H is sublinearly coercive if it satisfies (H1) for $m \leq 1$ and the following continuity condition holds:

(Ha) There exists a constant $C > 0$ and modulus of continuity ω_1 such that, for all $x, y, q, p \in \mathbb{R}^n$, we have

$$H(y,p) - H(x,q) \leq \omega_1(|x-y|)(1+|p|) + C(|p-q|).$$

Superlinear coercivity: We say that H is superlinearly coercive if:

(Hb) There exists $m > 1$,$A, \bar{C} > 0$ such that for all $\mu \in (0,1), x, y, p \in \mathbb{R}^n$, we have

$$H(x,p) - \mu H(x, \mu^{-1} p) \leq (1-\mu)\left(\bar{C}(1-m)|p|^m + A\right);$$

(Hc) If m is as in assumption (Hb), there exist $C > 0$ and a modulus of continuity ω_1 such that, for all $x, y, q, p \in \mathbb{R}^n$

$$H(y,p) - H(x,q) \leq \omega_1(|x-y|)(1+|p|^m \vee |q|^m) + C|p-q|(|p|^{m-1} \vee |q|^{m-1}).$$

Remark 4.2.4. *Note that condition* (Hb) *implies* $(H1)$ *for* $m > 1$.

As it is classical in viscosity solution's theory, the comparison principle allows the application of Perron's method to conclude the existence of solutions. To this end, we introduce the following assumption, which will allows us to build sub and supersolutions:

(E) There exists $H_R > 0$ such that for any $p \in \mathbb{R}^n, |p| \leq R$

$$||H(\cdot,p)||_\infty \leq H_R.$$

As a model example for sublinearly coercive Hamiltonian, we consider

$$H(x,p) = a_1(x)|p|^m + a_2(x)|p|^l - f(x),$$

with $m \leq 1, a_1 \geq a_0 > 0$ for all $x \in \Omega, l < m$ and $a_1, a_2, f : \bar{\Omega} \mapsto \mathbb{R}$ are continuous and bounded functions and a_1, a_2 are also Lipschitz continuous.
As a model example for superlinearly coercive Hamiltonian, we consider

$$H(x,p) = a_1(x)|p|^m + a_2(x)|p|^l + b(x) \cdot p - f(x),$$

with $m > 1, b$ bounded and continuous and a_1, a_2, f as before.
These Hamiltonians are coercive in p and in the case $m > 1$ we can include transport terms with a Lipschitz continuous vector field $b : \bar{\Omega} \mapsto \mathbb{R}^N$. The above assumptions are easily checkable in both cases.

Notion of viscosity solutions

We recall now the definition of solution to problem (4.4). We use the following notations: for any bounded function u

$$\mathscr{I}^\xi[u] = \int_{\substack{|z| \geq \xi, \\ x+j(x,z) \in \bar{\Omega}}} u(x+j(x,z)) - u(x) d\mu_x(z), \tag{4.15}$$

and for any C^1 function ϕ

$$\mathscr{I}_\xi[\phi] = \int_{\substack{|z|\le\xi,\\ x+j(x,z)\in\bar{\Omega}}} \phi(x+j(x,z)) - \phi(x)d\mu_x(z), \tag{4.16}$$

Note that the $\mathscr{I}^\xi$-term and the $\mathscr{I}_\xi$-term are well-defined respectively for u bounded and $\phi \in C^1$ thanks to (M0).

We also denote

$$F(x,u,Du,\mathscr{I}[u]) = u(x) - \mathscr{I}[u](x) + H(x,Du).$$

Following the approach of [24], we give the definition of viscosity solution to (4.4). Let C_j be defined as in (J1).

Definition 4.2.5. *(i) A bounded usc function u is a viscosity subsolution to* (4.4) *if, for any test-function $\phi \in C^1(\mathbb{R}^n)$ and maximum point x of $u-\phi$ in $\bar{B}_{C_j\xi}(x)\cap\bar{\Omega}$*

$$F(x,u(x),Du(x),\mathscr{I}_\xi[\phi]+\mathscr{I}^\xi[u]) \le 0 \qquad x \in \Omega$$

$$\min\{F(x,u(x),Du(x),\mathscr{I}_\xi[\phi]+\mathscr{I}^\xi[u]),\frac{\partial\phi}{\partial n}\} \le 0 \qquad x \in \partial\Omega.$$

(ii) A bounded lsc function v is a viscosity supersolution to (4.4) *if, for any test-function $\phi \in C^1(\mathbb{R}^n)$ and minimum point x of $v-\phi$ in $\bar{B}_{C_j\xi}(x)\cap\bar{\Omega}$,*

$$F(x,v(x),Dv(x),\mathscr{I}_\xi[\phi]+\mathscr{I}^\xi[v]) \ge 0 \qquad x \in \Omega$$

$$\max\{F(x,v(x),Dv(x),\mathscr{I}_\xi[\phi]+\mathscr{I}^\xi[v]),\frac{\partial\phi}{\partial n}\} \ge 0 \qquad x \in \partial\Omega.$$

(iii) A viscosity solution is both a sub- and a supersolution.

Main results

The main result of this part is the following comparison principle for the problem (4.4).

We recall that we denote by Hamiltonian of Bellman type an Hamiltonian defined as in (4.11) satisfying (C),(L), (B1), (B2) and by coercive Hamiltonian an Hamiltonian satisfying (H1) which can be either of sublinear type satisfying (Ha) or superlinear type satisfying (Hb) and (Hc).

Theorem 4.2.6. [Comparison] *Let Ω be an open subset of $\mathbb{R}^n$ satisfying (O). Assume $(M0),(M1),(J0),(J1),(J2)$. Let H be an Hamiltonian of Bellman type or a coercive Hamiltonian. Let u be a bounded usc subsolution of* (4.4) *and v a bounded lsc supersolution of* (4.4). *Then $u \le v$ in $\bar{\Omega}$.*

Once the comparison holds, we use the Perron's method for integro-differential equations (see [5], [27], [146] and [67],[110] for an introduction on the method) to get as a corollary existence and uniqueness for the problem (4.4) either when H is of Bellman type either when H is coercive.

Corollary 4.2.7. [Existence and Uniqueness] *Let Ω be an open subset of $\mathbb{R}^n$ satisfying (O). Assume $(M0),(M1),(J0),(J1),(J2)$ and let H be either an Hamiltonian of Bellman type or in coercive form satisfying (E).Then, there exists a unique bounded viscosity solution to problem* (4.4).

Proof. We construct constant sub and supersolutions thanks to assumption (E) taking respectively -$||H(\cdot,0)||_\infty$ and $||H(\cdot,0)||_\infty$. Then the proof follows by Theorem 4.2.6 and Perron's method. □

4.3 A preliminary key lemma

We prove the following Lemma 4.3.2, which is a key result used in the proof of Theorem 4.2.6. Roughly speaking, it deals with the difficulties arising from the way the geometry of the boundary interferes with the singularity of the nonlocal terms. The scope is to estimate the nonlocal terms defined in (4.18) on points near the boundary and equidistant from it. The approach of the proof is essentially based on a rectification of the boundary, relying on its regularity.

Remark 4.3.1. Note that in the case of domains with flat boundary, we do not need Lemma 4.3.2 in the proof of Theorem 4.2.6 since the estimation of the nonlocal terms can be carried out more easily. We refer to Remark 4.4.4, step 4 of the proof of Theorem 4.2.6.

Note that, if $\hat{s} \in \partial\Omega$, since Ω satisfies (O), there exists $r = r(\hat{s})$ and a $W^{2,\infty}$-diffeomorphism $\psi : B_r(\hat{s}) \mapsto \mathbb{R}^n$, satisfying

$$\psi_n(s) = d(s) \text{ for any } s \in B_r(\hat{s}), \tag{4.17}$$

where d is the signed distance from the boundary of Ω. For $s_1, s_2 \in B_{\frac{r}{2}}(\hat{s}) \cap \bar{\Omega}$, let

$$\mathscr{I}[J_{s_1}/J_{s_2}] = \int_{\substack{J_{s_1} \setminus J_{s_2},\\ |z| \le \delta_0}} \frac{dz}{|z|^{n+\sigma-1}} \tag{4.18}$$

where $J_s = \{z \in \mathbb{R}^n \,|\, s + j(s,z) \in \bar{\Omega}\}$, j satisfies assumptions (J0),(J1), (J2), $\sigma \in (0,1)$ and $0 < \delta_0 < rC_j^{-1}/2$, where C_j is the constant defined in (J1).

Lemma 4.3.2. *Let $\mathscr{I}[J./J.]$ as in* (4.18) *and assume j satisfies $(J0),(J1),(J2)$. Let $\hat{s} \in \partial\Omega$, r given as above and $s_1, s_2 \in \bar{\Omega}$ satisfying*

$$d(s_1) = d(s_2), \quad s_1, s_2 \in B_{\frac{r}{2}}(\hat{s}) \cap \bar{\Omega}. \tag{4.19}$$

Then there exists a positive constant C such that

$$\mathscr{I}[J_{s_1}/J_{s_2}] \le C|s_1 - s_2|. \tag{4.20}$$

Proof.

Step. *1-Rectification of the boundary* We observe that since $s_1, s_2 \in B_{\frac{r}{2}}(\hat{s}) \cap \bar{\Omega}$, $\delta_0 < rC_j^{-1}/2$ and by (J2), we have for any $|z| \le \delta_0$

$$s_1 + j(s_1,z),\, s_2 + j(s_2,z) \in B_r(\hat{s}). \tag{4.21}$$

By assumption (O), we describe the domain of integration of $\mathscr{I}[J_{s_1}/J_{s_2}]$ through the diffeomorphism ψ as follows

$$s_1 + j(s_1,z) \in \bar{\Omega} = \psi_n(s_1 + j(s_1,z)) \ge 0,$$

$$s_2 + j(s_2,z) \notin \bar{\Omega} = \psi_n(s_2 + j(s_2,z)) < 0.$$

We observe that by (4.17) and (4.19), we have

$$\psi_n(s_1) = \psi_n(s_2). \tag{4.22}$$

We proceed performing a change of variable in order to write the set of integration in terms of $\psi_n(s_1)$. In other words, we write

$$\psi(s_1 + j(s_1,z)) - \psi(s_1) = w, \tag{4.23}$$

that is, $j(s_1,z) = \psi^{-1}(\psi(s_1) + w) - s_1$. Then, the new set of integration can be written as follows

$$D = \{w \in \mathbb{R}^n : w_n + \psi_n(s_1) \ge 0,\, \psi_n(s_2 + j(s_2,z)) < 0,\, 0 < |w| \le \bar{C}\delta_0\}$$

In the following step, we rewrite D in a different way.

Step. *2-Rewriting the set D* By (4.23) and if $\psi_n(s_2 + j(s_2,z)) \le 0$, we have

$$\begin{aligned} w_n + \psi_n(s_1) &= \psi_n(s_2 + j(s_2,z)) + (\psi_n(s_1 + j(s_1,z)) - \psi_n(s_2 + j(s_2,z))) \\ &\le (\psi_n(s_1 + j(s_1,z)) - \psi_n(s_2 + j(s_2,z))). \end{aligned} \tag{4.24}$$

For convenience of notation, let for the moment

$$s(t) = ts_2 + (1-t)s_1, \quad j(t) = tj(s_2,z) + (1-t)j(s_1,z). \tag{4.25}$$

Note that $s(0)+j(0)=s_1+j(s_1,z)$, $s(1)+j(1)=s_2+j(s_2,z)$. Then, since $\psi\in W^{2,\infty}$ and by (4.24), we write

$$w_n+\psi_n(s_1)\le\int_0^1 D\psi_n(s(t)+j(t))\cdot(s_1+j(s_1,z)-(s_2+j(s_2,z))dt=A_1+A_2,$$

where

$$A_1=\int_0^1[D\psi_n(s(t)+j(t))-D\psi_n(s(t))]\cdot(s_1+j(s_1,z)-(s_2+j(s_2,z))dt,$$

$$A_2=\int_0^1 D\psi_n(s(t))\cdot(s_1-s_2)+\int_0^1 D\psi_n(s(t))\cdot(j(s_1,z)-j(s_2,z))dt.$$

From now on we denote by C any positive constant which may change from line to line. By definition of $j(t)$

$$|j(t)|=|tj(s_2,z)+(1-t)j(s_1,z)|\le 2C_j|z|\quad\text{for any } t\in[0,1]. \tag{4.26}$$

Then, since $\psi\in W^{2,\infty}$, by (4.26) and (J1), we get

$$A_1\le C\int_0^1|j(t)|(|s_1-s_2|+|j(s_1,z)-j(s_2,z)|)dt\le C|w||s_1-s_2|.$$

Now we analyse A_2. Note that by (4.25) and (4.22)

$$\int_0^1 D\psi_n(s(t))\cdot(s_1-s_2)=\int_0^1 D\psi_n((ts_2+(1-t)s_1)\cdot(s_1-s_2)=\psi_n(s_1)-\psi_n(s_2)=0.$$

Moreover, since $\psi\in W^{2,\infty}$ and by (J1)

$$\int_0^1 D\psi_n(s(t))\cdot(j(s_1,z)-j(s_2,z)))dt\le C|w||s_1-s_2|.$$

Then we have

$$A_2\le C|w||s_1-s_2|.$$

We denote $a=\psi_n(s_1)$ and observe $a\ge 0$. By all the previous arguments, we perform the change of variable in $\mathscr{I}[J_{s_1}/J_{s_2}]$ by (J0), (J1) and since $\psi\in W^{2,\infty}$ and we get for some constant $\bar{C}>0$

$$\mathscr{I}[J_{s_1}/J_{s_2}]\le\bar{C}\int_{\tilde{D}}\frac{dw}{|w|^{n+\sigma-1}}, \tag{4.27}$$

where

$$D\subset\tilde{D}=\{w\in\mathbb{R}^n : -a\le w_n\le -a+C|s_1-s_2||w|, 0<|w|\le\bar{C}\delta_0\}.$$

By no loss of generality and for simplicity of exposition, from now on we put $C=\bar{C}=1$.

Step. *3-Estimation on $\tilde{D}$* We introduce the following notations:

$$d = (1 - |s_1 - s_2|)^{-1} \quad \beta = (1 + |s_1 - s_2|)^{-1}. \tag{4.28}$$

Note that by the second assumption in (4.19), $|s_1 - s_2| \le r$. Without loss of generality we can suppose $r \le \frac{1}{2}$, so that we have $|s_1 - s_2| \le 1/2$. Then

$$2 \ge d \ge 1 \quad 1 \ge \beta \ge \frac{1}{2}. \tag{4.29}$$

Note that, if $w \in \tilde{D}$, then

$$-a \le w_n \le -a + |s_1 - s_2||w'| + |s_1 - s_2||w_n|. \tag{4.30}$$

We identify two cases, depending on the sign of $-a + |s_1 - s_2||w'|$ and we denote

$$D_1 = \left\{ w' \mid -a + |s_1 - s_2||w'| \ge 0, |w'| \le \delta_0 \right\}$$

and

$$D_2 = \left\{ w' \mid -a + |s_1 - s_2||w'| < 0, |w'| \le \delta_0 \right\}.$$

Observe that, if $w \in \tilde{D} \cap D_2$, then $-a + |s_1 - s_2||w'| < 0$ and (4.30) implies $w_n < 0$ and in particular

$$-a \le w_n \le -\beta a + \beta |s_1 - s_2||w'| < 0.$$

Otherwise, if $w \in \tilde{D} \cap D_1$, then $-a + |s_1 - s_2||w'| \ge 0$ and w_n can assume both negative and positive values. In particular (4.30) implies

$$-a \le w_n \le -da + d|s_1 - s_2|w'|.$$

Note also that $-da + d|s_1 - s_2|w'| \ge 0$. By all the previous observations, we write

$$\int_{\tilde{D}} \frac{dw}{|w|^{n+\sigma-1}} = \int_{\tilde{D}} \frac{dw_n dw'}{(|w'|^2 + |w_n|^2)^{\frac{n+\sigma-1}{2}}} \le \mathscr{F}_1 + \mathscr{F}_2, \tag{4.31}$$

where

$$\mathscr{F}_1 = \int_{D_1} \int_{-a}^{-da + d|s_1 - s_2||w'|} \frac{dw_n dw'}{(|w'|^2 + |w_n|^2)^{\frac{n+\sigma-1}{2}}},$$

$$\mathscr{F}_2 = \int_{D_2} \int_{-a}^{-\beta a + \beta |s_1 - s_2||w'|} \frac{dw_n dw'}{(|w'|^2 + |w_n|^2)^{\frac{n+\sigma-1}{2}}}.$$

For $\mathscr{F}_1$, we use that

$$\frac{1}{|w'|^2 + |w_n|^2} \le \frac{1}{|w'|^2}$$

and by Fubini's Theorem, we integrate in the n-variable and we get

$$\mathscr{F}_1 \leq \int_{D_1} \int_{-a}^{-da+d|s_1-s_2||w'|} \frac{dw_n dw'}{|w'|^{n+\sigma-1}} \leq \int_{D_1} \frac{-da+d|s_1-s_2||w'|+a}{|w'|^{n+\sigma-1}} dw'. \tag{4.32}$$

By the first of (4.28) and (4.29) and since $da \geq 0$, we have

$$-da+d|s_1-s_2||w'|+a = -da|s_1-s_2|+d|s_1-s_2||w'| \leq 2|s_1-s_2||w'|.$$

Therefore

$$\mathscr{F}_1 \leq d|s_1-s_2| \int_{D_1} \frac{dw'}{|w'|^{n+\sigma-2}}. \tag{4.33}$$

From now on we denote by C any positive constant which may change from line to line. Note that, since $w' \in \mathbb{R}^{n-1}$ and $\sigma < 1$, we have

$$\int_{D_1} \frac{dw'}{|w'|^{n+\sigma-2}} \leq C. \tag{4.34}$$

Then by the previous observations, we get

$$\mathscr{F}_1 \leq C|s_1-s_2|. \tag{4.35}$$

Now we analyse $\mathscr{F}_2$. For simplicity of notations, we denote

$$\zeta(w') = \int_{-a}^{-\beta a+\beta|s_1-s_2||w'|} \frac{dw_n}{(|w'|^2+w_n^2)^{\frac{n+\sigma-1}{2}}}$$

and then, by Fubini's Theorem, we have

$$\mathscr{F}_2 = \int_{D_2} \zeta(w')dw'. \tag{4.36}$$

We split the domain as follows

$$\int_{D_2} \zeta(w')dw = \int_{D_2\cap\{a\leq|w'|\}} \zeta(w')dw' + \int_{D_2\cap\{a>|w'|\}} \zeta(w')dw'. \tag{4.37}$$

We estimate the first term by

$$\int_{D_2\cap\{a\leq|w'|\}} \zeta(w')dw' \leq \int_{D_2\cap\{a\leq|w'|\}} \frac{-\beta a+\beta|s_1-s_2||w'|+a}{|w'|^{n+\sigma-1}} dw' \leq C|s_1-s_2|, \tag{4.38}$$

where in the first inequality we used that

$$-\beta a+\beta|s_1-s_2||w'|+a \leq 2|w'||s_1-s_2|,$$

since $\beta \leq 1$ and $a \leq |w'|$, and in the second inequality we used (4.34).

Take now the second term in (4.37). Note that, if $a > |w'|$, by (6.10) and (4.29), we have

$$-\beta a + \beta |s_1 - s_2||w'| \leq -\beta a d^{-1} \leq -a 4^{-1} \leq 0.$$

By all the previous observations, since the function $w_n \mapsto \frac{1}{\left(|w'|+w_n^2\right)^{\frac{n+\sigma-1}{2}}}$ is increasing on the negative halfline, we have

$$\zeta(w') \leq \frac{|s_1 - s_2|(a + |w'|)}{(|w'|^2 + 4^{-2}a^2)^{\frac{n+\sigma-1}{2}}} \leq 2^{n+\sigma-1}|s_1 - s_2| \frac{a + |w'|}{(|w'|^2 + a^2)^{\frac{n+\sigma-1}{2}}}. \tag{4.39}$$

Then

$$\int_{D_2 \cap \{|w'| \leq a\}} \frac{a + |w'|}{(|w'|^2 + a^2)^{\frac{n+\sigma-1}{2}}} \leq 2a \int_{D_2} \frac{dw'}{(|w',a|)^{n+\sigma-2}} \leq C \int_{D_2} \frac{dw'}{|w'|^{n+\sigma-2}} \leq C \tag{4.40}$$

and coupling (4.39) and (4.40), we get

$$\int_{D_2 \cap \{a \geq |w'|\}} \zeta(w') dw' \leq C|s_1 - s_2|. \tag{4.41}$$

Then coupling (4.41), (4.38), (4.37), (4.36), we obtain

$$\mathscr{F}_2 \leq C|s_1 - s_2| \tag{4.42}$$

and we conclude the proof by coupling (4.27), (4.31), (4.35) and (4.42).

□

4.4 Proof of the comparison principle

We prove Theorem 4.2.6 and we split the proof into two parts, depending whether H is of Bellman type or coercive.

4.4.1 Hamiltonians of Bellman type

We recall that we denote by Hamiltonian of Bellman type an Hamiltonian defined as in (4.11) satisfying (C),(L), (B1), (B2). The proof of Theorem 4.2.6 follows mainly by the following lemma, which we prove first. At the end of the proof of Lemma 4.2.6, we will prove Theorem 4.2.6.

Lemma 4.4.1. *Let Ω be an open subset of $\mathbb{R}^n$ satisfying (O). Let $\mathscr{I}$ as in* (4.5) *and assume μ satisfies $(M0), (M1)$, j satisfies $(J0), (J1), (J2)$. Let H be an Hamiltonian of Bellman type* *and let u, v be respectively bounded sub and supersolutions to* (4.4). *Then the function*

$$\omega(x) := u(x) - v(x)$$

satisfies, in the viscosity sense, the equation

$$\begin{cases} \omega - \mathscr{I}[\omega](x) - B|D\omega| \leq 0 & \text{in } \Omega \\ \frac{\partial \omega}{\partial n} = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.43}$$

where B is a positive constant depending on the data.

Proof. Let $x_0 \in \bar{\Omega}$ and $\phi \in C^1(\mathbb{R}^n)$ such that $\omega - \phi$ has a strict maximum point at x_0 . We observe that if $x_0 \in \Omega$ the proof is rather standard, since in this case the maximum points (x, y) of $u - v - \phi$ converge as $\varepsilon \to 0$ to (x_0, x_0) and hence they are bounded away from the boundary for ε small enough. This last property implies that we can directly use the equations and then proceed as in the following case.

Let $\Gamma_{\text{in}}, \Gamma_{\text{out}}, \Gamma$ be defined respectively in (4.12), (4.13) and (4.14) and recall they satisfy (B1) and (B2). We suppose $x_0 \in \partial\Omega$ and we split the proof depending if

(a) $x_0 \in \Gamma_{\text{in}}$;
(b) $x_0 \in \Gamma_{\text{out}}$;
(c) $x_0 \in \Gamma$.

In case (a) we use the existence of the blow-up supersolution which explodes at the boundary and allows us to keep the maximum points far from the boundary. Since the proof in this case is easier and is inspired by a similar approach used in [24] in the case of the halfspace (see Chapter 3 where we recall the approach of [24]), we give the details at the end of the proof of (b) and (c) in Remark 4.4.6.

Now we treat case (b) and (c). Since the proofs are similar, we treat them at the same time. Let $\varepsilon > 0$. We double the variable by introducing the function

$$\Phi(x, y) = u(x) - v(y) - \tilde{\phi}(x, y) \tag{4.44}$$

where

$$\tilde{\phi}(x, y) = \phi((x + y)/2) + \varepsilon^{-1}\chi_\varepsilon(|x - y|) + K\varepsilon^{-1}|d(x) - d(y)|, \tag{4.45}$$

where d is teh signed distance (see Remark 4.2.1), $\chi_\varepsilon : \mathbb{R} \to \mathbb{R}$ is defined as follows

$$\chi_\varepsilon(r) = \sqrt{r^2 + \varepsilon^4} \quad r \in \mathbb{R}, \tag{4.46}$$

$K > 0$ is a constant enough big such that

$$K > (2 + C_2)\gamma^{-1}, \tag{4.47}$$

where $\gamma, C_2 > 0$ depend on x_0 and are precisely defined in Lemma 5.3.9 in Appendix C (for $\hat{s} = x_0$). By upper-continuity, Φ attains its maximum over

$$A := \bar{B}_{2C_j}(x_0) \cap \bar{\Omega} \times \bar{B}_{2C_j}(x_0) \cap \bar{\Omega}$$

at a point (x,y). Moreover, by classical arguments in viscosity solution theory, we get as $\varepsilon \to 0$

$$x, y \to x_0, \quad \varepsilon^{-1}\chi_\varepsilon(|x-y|) \to 0, \quad \varepsilon^{-1}|d(x) - d(y)| \to 0 \tag{4.48}$$

and

$$u(x) - v(y) - \tilde{\phi}(x,y) \to u(x_0) - v(x_0) - \phi(x_0). \tag{4.49}$$

Then the function $\Phi(\cdot, y)$ has a local maximum point at x and $\Phi(x, \cdot)$ has a local minimum point at y. We suppose that

$$\frac{\partial \phi}{\partial n}(x_0) > 0, \tag{4.50}$$

then for ε small enough we have also

$$\frac{\partial \phi}{\partial n}((x+y)/2) > \frac{1}{2}\frac{\partial \phi}{\partial n}(x_0) > 0. \tag{4.51}$$

We observe that, if $d(x) = d(y)$, the test-function defined in (4.45) is not differentiable. In the following step we prove that this is the case (Lemma 4.4.2) and then we regularize the test function as showed in step 2.

Step. *1-Localising on equidistant points (i.e $d(x) = d(y)$)* We prove the following lemma. Note that the proof is slightly different in case (b) and case (c).

Lemma 4.4.2. *Under the above notations, we have*

$$d(x) = d(y).$$

Proof. We argue by contradiction and we suppose that $d(x) \neq d(y)$. First we prove that the F-viscosity inequalities (see Definition (4.2.5)) hold for u and v. Suppose that $x \in \partial\Omega$, then $d(x) = 0$ and $d(y) \neq 0$. We denote

$$\hat{p} = \frac{x-y}{|x-y|} \tag{4.52}$$

and we write

$$\frac{\partial \tilde{\phi}}{\partial n}(\cdot, y)(x) = \frac{1}{2}\frac{\partial \phi}{\partial n}((x+y)/2) + \varepsilon^{-1}\chi'_\varepsilon(|x-y|)\hat{p} \cdot n(x) + K\varepsilon^{-1} \tag{4.53}$$

Note that

$$0 \leq \chi'_\varepsilon(|x-y|) \leq 1. \tag{4.54}$$

Note that by (4.48), we can suppose, by taking ε small enough, that x, y belong to a suitable neighbourhood of the boundary where d is smooth (see Remark 4.2.1). Moreover, with some abuse of notations, we denote by $||D^2d||_\infty$ the supremum of $||D^2d||_\infty$ over this neighbourhood of the boundary. Then, by the Taylor's formula for the distance function, we have for ε small enough

$$n(x)(x-y)+\frac{1}{2}(x-y)^T D^2d(x)(x-y)+o(|x-y|^2)=d(y)\geq 0$$

and then

$$n(x)(x-y)\geq -||D^2d||_\infty|x-y|^2/2+o(|x-y|^2). \tag{4.55}$$

By (4.52), (4.54), (4.55) and (4.48), we have

$$\varepsilon^{-1}\chi'_\varepsilon(|x-y|)\hat{p}\cdot n(x)\geq o_\varepsilon(1). \tag{4.56}$$

By (4.53), (4.51), (4.56) and since $K\geq 0$, we conclude for ε small enough

$$\frac{\partial\tilde{\phi}}{\partial n}(\cdot,y)(x)\geq\frac{1}{4}\frac{\partial\phi}{\partial n}(x_0)+o_\varepsilon(1)+K\varepsilon^{-1}>0. \tag{4.57}$$

Then, since u is a viscosity subsolution and the function $u(\cdot)-v(y)-\tilde{\phi}(\cdot,y)$ has a local maximum at x, the F-viscosity inequality of Definition (4.2.5) (i) holds. A similar argument can be carried out for v.

From now on, we treat separately *Case* (b) ($x_0\in\Gamma_{\text{out}}$) and *Case* (c) ($x_0\in\Gamma$).

Case (b) In this case $x_0\in\Gamma_{\text{out}}$, where Γ_{out} is defined in (4.13). Suppose $d(x)>d(y)$. Then, for $2>\xi'>0$, by Definition (4.2.5) (i) and by (4.57), we have

$$u(x)-\mathscr{I}_{\xi'}[\tilde{\phi}(\cdot,y)](x)-\mathscr{I}^{\xi'}[u](x)+H(x,D[\tilde{\phi}(\cdot,y)](x))\leq 0. \tag{4.58}$$

Note that

$$D[\tilde{\phi}(\cdot,y)](x)=\varepsilon^{-1}(\chi'_\varepsilon(|x-y|)\hat{p}-Kn(x))+q,$$

where $\hat{p}$ is defined in (4.52) and

$$q=D\phi((x+y)/2)/2. \tag{4.59}$$

We apply Lemma 5.3.9, (5.32) with $\hat{s}=x_0$, $p=\varepsilon^{-1}\chi'_\varepsilon(|x-y|)\hat{p}+q$ and $\lambda=\varepsilon^{-1}K$ and by the definition (4.47) of K, we get for ε small

$$\begin{aligned}
H(x,D[\tilde{\phi}(\cdot,y)](x)) &\geq \varepsilon^{-1}\gamma K-C_2\left|\varepsilon^{-1}\chi'_\varepsilon(|x-y|)\hat{p}+q\right|-C_2\\
&\geq \varepsilon^{-1}(\gamma K-C_2)-C\\
&\geq \varepsilon^{-1}-C,
\end{aligned} \tag{4.60}$$

where by C, here and in the following, we denote any positive constant independent of ε which may change from line to line. In order to estimate the nonlocal terms we use the following lemma, which we prove in Appendix C.

Lemma 4.4.3. *Let* $\mathscr{I}^{\xi}, \mathscr{I}_{\xi}$ *be as in* (4.15)*,* (4.16) *and assume* $(M0)$*,* $(J1)$*. Under the same notations of Lemma 4.4.2, for any* $\xi > 0$*, there exist a positive constants* C_1 *independents of* ε *such that*

(i) $-\mathscr{I}_{\xi}[\tilde{\phi}(\cdot,y)](x) - \mathscr{I}^{\xi}[u](x) \geq -\varepsilon^{-1}C_1\xi^{1-\sigma} - C_1\xi^{-\sigma}$;

(ii) $-\mathscr{I}_{\xi}[\tilde{\phi}(x,\cdot)](y) - \mathscr{I}^{\xi}[v](y) \leq \varepsilon^{-1}C_1\xi^{1-\sigma} + C_1\xi^{-\sigma}$.

By (4.60), by Lemma 4.4.3 (i) with $\xi' = \varepsilon$ and by the boundedness of u, we write (4.58) as follows

$$-\varepsilon^{-\sigma} + \varepsilon^{-1} \leq C,$$

and we reach a contradiction for ε enough small since C is independent of ε and $\sigma < 1$. Now suppose $d(x) < d(y)$. In this case we use the following F-viscosity inequality for the supersolution v for $2 > \xi' > 0$

$$v(y) - \mathscr{I}_{\xi'}[-\tilde{\phi}(x,\cdot)](y) - \mathscr{I}^{\xi'}[v](y) + H(y, -D[\tilde{\phi}(x,\cdot)](y)) \geq 0 \tag{4.61}$$

We have

$$D[-\tilde{\phi}(\cdot,y)](x) = \varepsilon^{-1}(\chi'_{\varepsilon}(|x-y|)\hat{p} + Kn(y)) - q,$$

where $\hat{p}$ is defined in (4.52) and q is defined in (4.59). Then, for ε small enough, we apply Lemma 5.3.9, (5.33) with $\hat{s} = x_0$, $p = \varepsilon^{-1}\chi'_{\varepsilon}(|x-y|)\hat{p} - q$ and $\lambda = \varepsilon^{-1}K$ and by the definition of K (4.47) we get

$$\begin{aligned} H(y, -D[\tilde{\phi}(x,\cdot)](y)) &\leq -\varepsilon^{-1}\gamma K + C_2\left|\varepsilon^{-1}\chi'_{\varepsilon}(|x-y|)\hat{p} - q\right| \\ &\leq -\varepsilon^{-1}(\gamma K + C_2) + C \\ &\leq -\varepsilon^{-1} - C. \end{aligned} \tag{4.62}$$

We proceed as in the previous case, by applying Lemma 4.4.3 (ii) with $\xi = \varepsilon$ and by (4.62) and the boundedness of v , we get

$$\varepsilon^{-\sigma} - \varepsilon^{-1} \geq C$$

and we reach a contradiction for ε small enough as above.

Case (c) In this case $x_0 \in \Gamma$, where Γ is defined in (4.14). If $d(x) > d(y)$ the proof is the same. If $d(x) < d(y)$ we write again equation (4.58) and since

$$D[\tilde{\phi}(\cdot,y)](x) = \varepsilon^{-1}(\chi'_{\varepsilon}(|x-y|)\hat{p} + Kn(x)) + q,$$

where $\hat{p}$ is defined in (4.52), we apply Lemma 5.3.9, (5.35) with $\hat{s} = x_0$, $p = \varepsilon^{-1}\chi'_\varepsilon(|x - y|)\hat{p}) + q$ and $\lambda = \varepsilon^{-1}K$ (for ε small enough small) and we conclude as above. □

Step. *2-Regularizing the test-function* Since the test function defined in (4.45) is not differentiable on the points where $d(x) = d(y)$, we regularize it as follows:

$$\tilde{\phi}(x,y) = \phi((x+y)/2) + \varepsilon^{-1}\chi_\varepsilon(|x-y|) + K\varepsilon^{-1}\chi_\delta(|d(x) - d(y)|) \tag{4.63}$$

where d is the signed distance from the boundary (see Reamrk 4.2.1) and $\chi_\varepsilon, \chi_\delta$ are as in (4.46). We use the same notation as before

$$\Phi(x,y) = u(x) - v(y) - \tilde{\phi}(x,y) \tag{4.64}$$

and denote by $(\bar{x},\bar{y})$ the maximum point of Φ in $\big(\bar{B}_{2C_j}(x_0) \cap \bar{\Omega}\big) \times \big(\bar{B}_{2C_j}(x_0) \cap \bar{\Omega}\big)$. We observe that $(\bar{x},\bar{y})$ depends now also on δ and we omit the dependence. Using classical arguments, we get that as $\delta \to 0$

$$\bar{x} \to x, \quad \bar{y} \to y, \quad u(\bar{x}) \to u(x), \quad v(\bar{y}) \to v(y), \tag{4.65}$$

where (x,y) is a maximum point of the function defined in (4.44) in $\big(\bar{B}_{2C_j}(x_0) \cap \bar{\Omega}\big) \times \big(\bar{B}_{2C_j}(x_0) \cap \bar{\Omega}\big)$. Since we proved in Step 1 that $d(x) = d(y)$, we have, as $\delta \to 0$,

$$d(\bar{x}) - d(\bar{y}) \to 0. \tag{4.66}$$

To fix the ideas, from now on we consider δ, ε small enough so that

$$\bar{x}, \bar{y}, x, y \in B_{C_j}(x_0). \tag{4.67}$$

Now we prove that the F-viscosity inequalities for u and v hold. We take $\bar{x} \in \partial\Omega$ and we show that the boundary conditions do not hold, so the F-viscosity inequalities hold as in Definition (4.2.5). We proceed exactly as in Step 1, Lemma 4.4.2, so we omit the details. We recall that

$$0 \le \chi'_\delta(|x-y|) \le 1, \quad \text{for all } x,y \in \bar{\Omega}, \tag{4.68}$$

and we note only that (4.57) now reads for ε, δ small enough

$$\frac{\partial \tilde{\phi}}{\partial n}(\cdot,\bar{y})(\bar{x}) \ge \frac{1}{4}\frac{\partial \phi}{\partial n}(x_0) + o_{\delta,\varepsilon}(1) + K\varepsilon^{-1}\chi'_\delta(d(\bar{y})) > 0,$$

since $d(\bar{x}) = 0$, $\chi'_\delta(d(\bar{y})) \ge 0$ and $o_{\delta,\varepsilon}(1)$ means that $\lim_{\delta\to 0} o_{\delta,\varepsilon}(1) = o_\varepsilon(1)$. Then for $1 > \xi' > 0$, we have

$$\begin{aligned} u(\bar{x}) - v(\bar{y}) \;&\le\; H(\bar{y}, -D[\tilde{\phi}(\bar{x},\cdot)](\bar{y})) - H(\bar{x}, D[\tilde{\phi}(\cdot,\bar{y})](\bar{x})) \\ &+\; \mathscr{I}^{\xi'}[u](\bar{x}) - \mathscr{I}^{\xi'}[v](\bar{y}) + \mathscr{I}_{\xi'}[\tilde{\phi}(\cdot,y)](\bar{x}) - \mathscr{I}_{\xi'}[-\tilde{\phi}(\bar{x},\cdot)](\bar{y}). \end{aligned} \tag{4.69}$$

Since $\tilde{\phi} \in C^1$, we have

$$\mathscr{I}_{\xi'}[\tilde{\phi}(\cdot,y)](x) \leq C_j ||D\tilde{\phi}||_{L^\infty(\bar{B}(0,C_j\xi'))} \int_{\mathbb{R}^n} 1_{|z|\leq\xi'} |z| d\mu_x(z) = o_{\xi'}(1). \tag{4.70}$$

where we used (J1) and (M0), C_j is as in (J1) and $o_{\xi'}(1)$ is independent of δ. The same holds for $-\mathscr{I}_{\xi'}[-\tilde{\phi}(\bar{x},\cdot)](\bar{y})$.
Note that

$$\left|D[\tilde{\phi}(\cdot,\bar{y})](\bar{x}) - D[-\tilde{\phi}(\bar{x},\cdot)](\bar{y})\right| = \varepsilon^{-1}\left|K\chi_\delta'(|d(\bar{x})-d(\bar{y})|)\tilde{p}\,(n(\bar{y})-n(\bar{x}))\right| + |D\phi((\bar{x}+\bar{y})/2)|, \tag{4.71}$$

where

$$\tilde{p} = \frac{d(\bar{x})-d(\bar{y})}{|d(\bar{x})-d(\bar{y})|}. \tag{4.72}$$

We suppose ε,δ small enough so that $\bar{x},\bar{y}$ belong to the neighbourhood of the boundary where the distance is smooth (see Remark 4.2.1). By the smoothness of the distance function we have

$$\begin{aligned} \left|D[\tilde{\phi}(\cdot,\bar{y})](\bar{x}) - D[-\tilde{\phi}(\bar{x},\cdot)](\bar{y})\right| &\leq \varepsilon^{-1}K|n(\bar{y})-n(\bar{x})| + |D\phi((\bar{x}+\bar{y})/2)| \\ &\leq \varepsilon^{-1}K|\bar{x}-\bar{y}| + |D\phi((\bar{x}+\bar{y})/2)|. \end{aligned} \tag{4.73}$$

By the definition of H and (4.73), we have

$$H(\bar{y},-D[\tilde{\phi}(\bar{x},\cdot)](\bar{y})) - H(\bar{y},D[\tilde{\phi}(\cdot,\bar{y})](\bar{x}) \leq B\left(|D\phi((\bar{x}+\bar{y})/2)| + K\varepsilon^{-1}|\bar{x}-\bar{y}|\right), \tag{4.74}$$

where $B = \sup_{x\in\bar{\Omega},\alpha\in\mathscr{A}} b(x,\alpha)$. Moreover by $(C),(L)$, we have

$$H(\bar{y},D[\tilde{\phi}(\cdot,\bar{y})](\bar{x}) - H(\bar{x},D[\tilde{\phi}(\cdot,\bar{y})](\bar{x})) \leq B|\bar{x}-\bar{y}||D[\tilde{\phi}(\cdot,\bar{y})](\bar{x})| + \omega_l(|\bar{x}-\bar{y}|)$$

and since

$$|D[\tilde{\phi}(\cdot,\bar{y})](\bar{x})| \leq K\varepsilon^{-1} + 2^{-1}||D\phi||_{L^\infty(B_{2C_j}(x_0))}, \tag{4.75}$$

we get

$$H(\bar{y},D[\tilde{\phi}(\cdot,\bar{y})](\bar{x}) - H(\bar{x},D[\tilde{\phi}(\cdot,\bar{y})](\bar{x})) \leq C\left(\varepsilon^{-1}|\bar{x}-\bar{y}| + |\bar{x}-\bar{y}|\right) + \omega_l(|\bar{x}-\bar{y}|), \tag{4.76}$$

where $C > 0$ is a constant depending on B,K and $||D\phi||_{L^\infty(B_{2C_j}(x_0))}$. By coupling (4.74) and (4.76) and by (4.48) and (4.65), we get

$$H(\bar{y},-D[\tilde{\phi}(\bar{x},\cdot)](\bar{y})) - H(\bar{x},D[\tilde{\phi}(\cdot,\bar{y})](\bar{x}) \leq B|D\phi((\bar{x}+\bar{y})/2)| + o_{\delta,\varepsilon}(1), \tag{4.77}$$

where $o_{\delta,\varepsilon}(1)$ means $\lim_{\delta\to 0} o_{\delta,\varepsilon}(1) = o_\varepsilon(1)$. Plugging (4.77) and (4.70) into (4.69), we get

$$u(\bar{x}) - v(\bar{y}) \leq B|D\phi((\bar{x}+\bar{y})/2)| + \mathscr{I}^{\xi'}[u](\bar{x}) - \mathscr{I}^{\xi'}[v](\bar{y}) + o_{\delta,\varepsilon}(1) + o_{\xi'}(1). \tag{4.78}$$

We want to send first $\delta \to 0$ in (4.78) and we observe that the nonlocal terms are uniformly bounded in δ. Consider $\mathscr{I}^{\xi'}[u](\bar{x})$, observing that the same argument works similarly for $\mathscr{I}^{\xi'}[v](\bar{y})$. Note that by (4.67) and (J1), if $|z| < 1$, then $\bar{x} + j(\bar{x},z) \in B_{2C_j}(x_0)$. Since $(\bar{x},\bar{y})$ is a maximum point on $\left(\bar{B}_{2C_j}(x_0) \cap \bar{\Omega}\right) \times \left(\bar{B}_{2C_j}(x_0) \cap \bar{\Omega}\right)$ of Φ defined in (4.64), we have for δ, ε small

$$u(\bar{x} + j(\bar{x},z)) - u(\bar{x}) = u(\bar{x} + j(\bar{x},z)) - v(\bar{y}) - (u(\bar{x}) - v(\bar{y})) \le \tilde{\phi}(\bar{x} + j(\bar{x},z),\bar{y}) - \tilde{\phi}(\bar{x},\bar{y}).$$

Note that χ_δ is Lipschitz with Lipschitz constant independent of δ thanks to (4.68). Then, by the definition of $\tilde{\phi}$, since $\chi_\varepsilon, \chi_\delta, \phi$ are Lipschitz and by (J1), we have

$$u(\bar{x} + j(\bar{x},z)) - u(\bar{x}) \le C\varepsilon^{-1}|z| + C|z| \tag{4.79}$$

which by (M0) gives the uniform boundedness in δ of $\mathscr{I}^{\xi'}[u](\bar{x})$ when $|z| < 1$. When $|z| \ge 1$, the claim simply follows by the boundedness of u and (M0).

Then, we send $\delta \to 0$ in (4.78) and we apply Fatou's Lemma. By the semicontinuity and boundedness of u and v , we get

$$u(x) - v(y) \le B|D\phi((x+y)/2)| + \mathscr{I}^{\xi'}[u](x) - \mathscr{I}^{\xi'}[v](y) + o_\varepsilon(1) + o_{\xi'}(1). \tag{4.80}$$

Now we analyse the term $\mathscr{I}^{\xi'}[u](x) - \mathscr{I}^{\xi'}[v](y)$. We observe that, for simplicity of exposition, we first conclude the proof in the case the measure μ in the nonlocal terms has no dependence on x, i.e. $\mu_x \equiv \mu$. We refer to Remark 4.4.5 for details in the case of x-dependence. We write

$$\mathscr{I}^{\xi'}[u](x) - \mathscr{I}^{\xi'}[v](y) = I^{\xi'}[J_x/J_y] + I^{\xi'}[J_y/J_x] + T^{\xi'}[J_x \cap J_y], \tag{4.81}$$

where

$$J_x = \{z \in \mathbb{R}^n \,|\, x + j(x,z) \in \bar{\Omega}\} \tag{4.82}$$

and

$$I^{\xi'}[J_x/J_y] = \int_{\substack{J_x/J_y,\\ |z| \ge \xi'}} u(x + j(x,z)) - u(x)d\mu(z); \tag{4.83}$$

$$I^{\xi'}[J_y/J_x] = \int_{\substack{J_y/J_x,\\ |z| \ge \xi'}} v(y) - v(y + j(y,z))d\mu(z);$$

$$T^{\xi'}[J_x \cap J_y] = \int_{\substack{J_x \cap J_y,\\ |z| \ge \xi'}} [u(x + j(x,z)) - u(x) - (v(y + j(y,z)) - v(y))]d\mu(z).$$

Consider $T^{\xi'}[J_x \cap J_y]$. Recall that $(\bar{x},\bar{y})$ satisfy for any $x',y' \in \left(\bar{B}_{2C_j}(x_0)\cap\bar{\Omega}\right)\times\left(\bar{B}_{2C_j}(x_0)\cap\bar{\Omega}\right)$

$$u(\bar{x})-v(\bar{y})-\tilde{\phi}(\bar{x},\bar{y}) \geq u(x')-v(y')-\tilde{\phi}(x',y'). \tag{4.84}$$

Letting $\delta \to 0$ in (4.84), by (4.65), (4.66), the definition of $\tilde{\phi}$ and the semicontinuity of u,v, we get for any $x',y' \in \left(\bar{B}_{2C_j}(x_0)\cap\bar{\Omega}\right)\times\left(\bar{B}_{2C_j}(x_0)\cap\bar{\Omega}\right)$

$$\begin{aligned} u(x')-u(x)-(v(y')-v(y)) \leq \varepsilon^{-1}\chi_\varepsilon(|x'-y'|)-\varepsilon^{-1}\chi_\varepsilon(|x-y|) \\ +\phi((x'+y'/2)-\phi((x+y)/2). \end{aligned} \tag{4.85}$$

If $|z|<1$, by (4.67) and (J1), $x+j(x,z), y+j(y,z) \in B_{2C_j}(x_0)$. Then we write (4.85) for $x'=x+j(x,z), y'=y+j(y,z)$ and we have

$$\begin{aligned} u(x+j(x,z))-u(x) \quad &- \quad (v(y+j(y,z))-v(y)) \\ &\leq \quad \varepsilon^{-1}\chi_\varepsilon(|x+j(x,z)-y-j(y,z)|)-\varepsilon^{-1}\chi_\varepsilon(|x-y|) \\ &+ \quad \phi((x+j(x,z)+y+j(y,z))/2)-\phi((x+y)/2). \end{aligned}$$

Note that by the Lipschitz continuity of χ_ε, (J2) and (4.48), we have

$$\varepsilon^{-1}\chi_\varepsilon(|x+j(x,z)-y-j(y,z)|)-\varepsilon^{-1}\chi_\varepsilon(|x-y|) \leq D_j|z|\varepsilon^{-1}|x-y| = |z|o_\varepsilon(1) \tag{4.86}$$

where D_j is defined in (J2) and then

$$\begin{aligned} u(x+j(x,z))-u(x)-(v(y+j(y,z))-v(y)) \\ \leq \phi((x+j(x,z)+y+j(y,z))/2)-\phi((x+y)/2)+|z|o_\varepsilon(1). \end{aligned} \tag{4.87}$$

Then for $0<\xi'<\xi<1$, by (4.87) and (M0), we get

$$T^{\xi'}[J_x \cap J_y] \leq P_\xi - P_{\xi'} + K^\xi + o_\varepsilon(1), \tag{4.88}$$

where $o_\varepsilon(1)$ is independent of ξ' and

$$K^\xi = \int_{\substack{J_x\cap J_y,\\ |z|\geq\xi}} u(x+j(x,z))-u(x)-(v(y+j(y,z))-v(y))d\mu(z),$$

$$P_{\xi'} = \int_{\substack{J_x\cap J_y,\\ |z|\leq\xi'}} \phi((x+j(x,z)+y+j(y,z))/2)-\phi((x+y)/2)d\mu(z),$$

$$P_\xi = \int_{\substack{J_x\cap J_y,\\ |z|\leq\xi}} \phi((x+j(x,z)+y+j(y,z))/2)-\phi((x+y)/2)d\mu(z).$$

Since ϕ is Lipschitz, by (J1) and (M0), we have

$$P_{\xi'} = o_{\xi'}(1). \tag{4.89}$$

Now we consider the term $I^{\xi'}[J_x/J_y]$ (defined in (4.83)), observing that the same argument works similarly for $I^{\xi'}[J_y/J_x]$. Take $0 < \delta_0 < 1$ enough small (note that δ_0 will be defined more precisely in Step 4). We split the domain of integration into $\{z : |z| \geq \delta_0\}$ and $\{z : \xi' \leq |z| < \delta_0\}$. We write

$$I^{\xi'}[J_x/J_y] = I^{\xi'}[B^c_{\delta_0}] + I^{\xi'}[B_{\delta_0}] \tag{4.90}$$

where

$$I^{\xi'}[B^c_{\delta_0}] = \int_{\substack{J_x/J_y,\\ |z|\geq\delta_0}} u(x+j(x,z)) - u(x) d\mu(z)$$

$$I^{\xi'}[B_{\delta_0}] = \int_{\substack{J_x/J_y,\\ \xi'\leq|z|<\delta_0}} u(x+j(x,z)) - u(x) d\mu(z).$$

By the boundedness of u, we have

$$I^{\xi'}[B^c_{\delta_0}] \leq 2C||u||_\infty \int_{|z|\geq\delta_0} 1_{J_x/J_y} d\mu(z)$$

and since

$$|J_x/J_y| \to 0 \text{ as } \varepsilon \to 0, \tag{4.91}$$

by (M0), the Dominated Convergence theorem and (4.91), we get

$$I^{\xi'}[B^c_{\delta_0}] \leq o_\varepsilon(1), \tag{4.92}$$

where $o_\varepsilon(1)$ is independent of ξ'.

For $I^{\xi'}[B_{\delta_0}]$ we use again the maximum point inequality (4.85) with $x' = x + j(x,z), y' = y$ and since $\phi \in C^1$ and by (J1), (M0), we get

$$I^{\xi'}[B_{\delta_0}] \leq C\varepsilon^{-1} \int_{\substack{J_x/J_y,\\ \xi'\leq|z|\leq\delta_0}} \frac{dz}{|z|^{N+\sigma-1}}, \tag{4.93}$$

where we remark that $C > 0$ is independent of all the parameters.

Observe that the integral in (4.93) is uniformly bounded in ξ' and then we can send $\xi' \to 0$ by the Dominated Convergence Theorem. We couple (4.81), (4.88), (4.89), (4.90), (4.92) and (4.93) with (4.80) and we send $\xi' \to 0$ getting

$$\begin{aligned} u(x)-v(y) &- B|D\phi((x+y)/2)| \\ &\leq C\varepsilon^{-1}I[J_x/J_y]+C\varepsilon^{-1}I[J_y/J_x]+P_\xi+K^\xi+o_\varepsilon(1), \end{aligned} \tag{4.94}$$

where for all $x,y \in \mathbb{R}^n$, we denote

$$I[J_x/J_y] := \int_{\substack{J_x/J_y, \\ |z|\leq\delta_0}} \frac{dz}{|z|^{N+\sigma-1}}. \tag{4.95}$$

Step. *4-Estimation of the term* (4.95) *by Lemma 4.3.2*
Let $r := r(x_0)$, where $r(x_0)$ is defined in assumption (O) for $\hat{s} = x_0$. Take $rC_j^{-1}/2 > \delta_0$. Note that, by (4.67), (x,y) satisfy (4.19) for $\hat{s} = x_0$ and $r = r(x_0)$. Then we apply Lemma 4.3.2 by taking $\{s_1,s_2\} = \{x,y\}, \hat{s} = x_0$ in order to estimate $\mathscr{I}[J_x/J_y], \mathscr{I}[J_y/J_x]$ defined in (4.95) and we get

$$\mathscr{I}[J_x/J_y] \leq C|x-y|, \quad \mathscr{I}[J_y/J_x] \leq C|x-y|. \tag{4.96}$$

Note that Lemma 4.3.2 is not necessary when dealing with domains with flat boundary. In the following remark we consider the case when Ω is the halfspace and we show how the estimation of the nonlocal terms can be carried out more easily without Lemma 4.3.2.

Remark 4.4.4. We consider the case when Ω is the half-space, i.e.

$$\Omega := \{(x_1,\cdots,x_n = (x',x_n) \in \mathbb{R}^n : x_n > 0\}. \tag{4.97}$$

For simplicity, we suppose that $j(x,z) = z$ if $x+z \in \bar{\Omega}$. Note that (4.66) reads

$$\bar{x}_n - \bar{y}_n \to 0 \quad \text{as } \delta \to 0. \tag{4.98}$$

We recall the equation (4.69)

$$\begin{aligned} u(\bar{x})-v(\bar{y}) &\leq H(\bar{y},-D[\tilde{\phi}(\bar{x},\cdot)](\bar{y}))-H(\bar{x},D[\tilde{\phi}(\cdot,\bar{y})](\bar{x}) \\ &+ \mathscr{I}^{\xi'}[u](\bar{x})-\mathscr{I}^{\xi'}[v](\bar{y})+\mathscr{I}_{\xi'}[\tilde{\phi}(\cdot,y)](\bar{x})-\mathscr{I}_{\xi'}[-\tilde{\phi}(\bar{x},\cdot)](\bar{y}), \end{aligned} \tag{4.99}$$

where now

$$\mathscr{I}^{\xi'}[u](\bar{x}) = \int_{\substack{\bar{x}_n+z_n\geq 0, \\ |z|\geq\xi'}} u(\bar{x}+z)-u(\bar{x})d\mu(z)$$

and

$$\mathscr{I}^{\xi'}[v](\bar{y}) = \int_{\substack{\bar{y}_n+z_n\geq 0, \\ |z|\geq\xi'}} v(\bar{y}+z)-v(\bar{y})d\mu(z).$$

The estimations of the $\mathscr{I}_\xi$-terms and the Hamiltonian terms are the same as in the non flat case (see the previous step). Consider the nonlocal terms in (4.99) and restrict ourselves to a subsequence such that $\bar{x}_n \geq \bar{y}_n$ (if $\bar{x}_n \leq \bar{y}_n$ the argument is similar). Then we can write

$$\begin{aligned}
\mathscr{I}^{\xi'}[u](\bar{x}) - \mathscr{I}^{\xi'}[v](\bar{y}) &= \int_{\substack{-\bar{x}_n \leq z_n < -\bar{y}_n, \\ |z| \geq \xi'}} [u(\bar{x}+z) - u(\bar{x})]d\mu_{\bar{x}}(z) \\
&+ \int_{\substack{-\bar{y}_n \leq z_n, \\ |z| \geq \xi'}} [u(\bar{x}+z) - v(\bar{y}+(j(\bar{y},z)) - (u(\bar{x}) - v(\bar{y}))]d\mu_{\bar{x}}(z).
\end{aligned}$$

For coherence with the notations used in the non flat case, we denote

$$J_s := \{z \in \mathbb{R}^n \,|\, s_n + z_n \geq 0\},$$

$$I^{\xi'}[J_{\bar{x}}/J_{\bar{y}}] := \int_{\substack{-\bar{x}_n \leq z_n < -\bar{y}_n, \\ |z| \geq \xi'}} [u(\bar{x}+z) - u(\bar{x})]d\mu_{\bar{x}}(z)$$

$$T^{\xi'}[J_{\bar{x}} \cap J_{\bar{y}}] := \int_{\substack{-\bar{y}_n \leq z_n, \\ |z| \geq \xi'}} [u(\bar{x}+z) - v(\bar{y}+(j(\bar{y},z)) - (u(\bar{x}) - v(\bar{y}))]d\mu_{\bar{x}}(z).$$

Then

$$\mathscr{I}^{\xi'}[u](\bar{x}) - \mathscr{I}^{\xi'}[v](\bar{y}) \leq I^{\xi'}[J_{\bar{x}}/J_{\bar{y}}] + T^{\xi'}[J_{\bar{x}} \cap J_{\bar{y}}].$$

The term $T^{\xi'}[J_{\bar{x}} \cap J_{\bar{y}}]$ is treated exactly as in the non flat case (see the previous step). On the contrary, note that in this case the estimation of the term $I^{\xi'}[J_{\bar{x}}/J_{\bar{y}}]$ is easier, since by (4.98)

$$|J_{\bar{x}}/J_{\bar{y}}| \to 0 \quad \text{as } \delta \to 0$$

and then by the Dominated Convergence Theorem, we have

$$I^{\xi'}[J_{\bar{x}}/J_{\bar{y}}] \to 0 \quad \text{as } \delta \to 0.$$

Step. *5-Sending the parameters to their limits*

We plug (4.96) into (4.94) and we get

$$u(x) - v(y) - B|D\phi((x+y)/2)| \leq C\varepsilon^{-1}|x-y| + P_\xi + K^\xi + o_\varepsilon(1) \tag{4.100}$$

where C is a constant independent of ξ. Moreover, since ϕ is C^1, by (M0), the Dominated Convergence Theorem and since $x, y \to x_0$ as $\varepsilon \to 0$, we have

$$\limsup_{\varepsilon \to 0} P_\xi \leq \mathscr{I}_\xi[\phi](x_0) \tag{4.101}$$

and by the boundedness and semicontinuity of u, v and applying Fatou's lemma for each $\xi > 0$ fixed, we have

$$\limsup_{\varepsilon \to 0} K^{\xi} \leq \mathscr{I}^{\xi}[\omega(\cdot, t_0)](x_0), \tag{4.102}$$

and we conclude by sending $\varepsilon \to 0$ in (4.100).

□

Remark 4.4.5. We give some details of the analysis of the nonlocal terms in step 2 when the measure μ depends on x. In this case we write (4.81) with

$$I^{\xi'}[J_x/J_y] = \int_{\substack{J_x/J_y, \\ |z| \geq \xi'}} u(x + j(x,z)) - u(x) d\mu_x(z);$$

$$I^{\xi'}[J_y/J_x] = \int_{\substack{J_y/J_x, \\ |z| \geq \xi'}} v(y) - v(y + j(y,z)) d\mu_y(z);$$

$$T^{\xi'}[J_x \cap J_y] = \int_{\substack{J_x \cap J_y, \\ |z| \geq \xi'}} [u(x + j(x,z)) - u(x)] d\mu_x(z) - [(v(y + j(y,z)) - v(y))] d\mu_y(z).$$

For $I^{\xi'}[J_x/J_y]$ and $I^{\xi'}[J_y/J_x]$ we proceed exactly as showed above in step 2, observing that the x-dependence does not play any role by (M0). For the T-term, we write

$$T^{\xi'}[J_x \cap J_y] = T_1^{\xi'}[J_x \cap J_y] + T_2^{\xi'}[J_x \cap J_y]$$

where

$$T_1^{\xi'}[J_x \cap J_y] = \int_{\substack{J_x \cap J_y, \\ |z| \geq \xi'}} u(x + j(x,z)) - u(x) - (v(y + j(y,z)) - v(y)) d\mu_y(z)$$

$$T_2^{\xi'}[J_x \cap J_y] = \int_{\substack{J_x \cap J_y, \\ |z| \geq \xi'}} [(u(x + j(x,z)) - u(x))](d\mu_x(z) - d\mu_y(z))$$

For $T_1^{\xi'}[J_x \cap J_y]$, we proceed as showed above for $T^{\xi'}[J_x \cap J_y]$ and we prove (4.88). Now consider $T_2^{\xi'}[J_x \cup J_y]$. Take $0 < \xi' < \xi < 1$ and denote

$$T_2^{\xi'}[J_x \cap J_y] = T_2^{\xi'}[B_\xi] + T_2^{\xi'}[B_\xi^c]$$

where

$$T_2^{\xi'}[B_\xi] = \int_{\substack{J_x \cap J_y, \\ \xi \geq |z| \geq \xi'}} [(u(x + j(x,z)) - u(x))](d\mu_x(z) - d\mu_y(z))$$

$$T_2^{\xi'}[B_\xi^c] = \int_{\substack{J_x \cap J_y, \\ |z| > \xi}} [(u(x+j(x,z)) - u(x))](d\mu_x(z) - d\mu_y(z)).$$

For $T_2^{\xi'}[B_\xi]$ we use again the maximum point inequality (4.85) and we write for $|z| \le \xi$

$$\begin{aligned} u(x+j(x,z)) - u(x) \le \varepsilon^{-1}\chi_\varepsilon(|x+j(x,z)-y|) - \varepsilon^{-1}\chi_\varepsilon(|x-y|) \\ + \phi((x+j(x,z)+y)/2) - \phi(x+y)/2. \quad (4.103) \end{aligned}$$

Then by the Lipschitz continuity of χ_ε and ϕ, (J1), (M0), (M1) and (4.48) we get

$$T_2^{\xi'}[B_\xi] \le C \int_{\substack{J_x \cap J_y, \\ \xi \ge |z| \ge \xi'}} (\varepsilon^{-1}|z| + |z|)(d\mu_x(z) - d\mu_y(z)) \le o_\varepsilon(1), \quad (4.104)$$

where we observe $o_\varepsilon(1)$ is independent of ξ' and may change from line to line in the following. For $T_2^{\xi'}[B_\xi^c]$, we use the boundedness of u and by (M0), (M1) and (4.48), we write

$$T_2^{\xi'}[B_\xi^c] \le 2||u||_\infty \int_{\substack{J_x \cap J_y, \\ |z| > \xi}} (d\mu_x(z) - d\mu_y(z)) \le o_\varepsilon(1). \quad (4.105)$$

Then, by (4.104) and (4.105), we get

$$T_2^{\xi'}[J_x \cap J_y] \le o_\varepsilon(1), \quad (4.106)$$

where we observe $o_\varepsilon(1)$ is independent of ξ'. From now on the proof is the same as above.

Remark 4.4.6. We give the details of the proof of Lemma 4.4.1 in case (a), when $x_0 \in \Gamma_{\text{in}}$ is a strict maximum point of $\omega - \phi = u - v - \phi$, where $\phi \in C^1(\mathbb{R}^n)$. The strategy of the proof relies on the existence of a blow-up supersolution exploding on the boundary, which allows us to keep the maximum points away from the boundary. The existence of such a supersolution is stated in the following lemma, whose proof is given in Appendix C.

Lemma 4.4.7. *For any $\bar{x} \in \Gamma_{in}$, there exists $r = r(\bar{x}) > 0$ and a positive function $U_r \in C^2(B_r(\bar{x}) \cap \Omega)$ satisfying for any ξ small enough (with respect to r, that is $\xi < C_j^{-1}\frac{r}{2}$)*

(i)

$$-b(x,\alpha)\cdot DU_r - \mathscr{I}_\xi[U_r](x) \ge 0 \quad \text{in } B_{\frac{r}{2}}(\bar{x}) \cap \Omega, \quad \forall \alpha \in \mathscr{A};$$

(ii)

$$U_r(x) \ge \frac{1}{\omega_r(d(x))} \quad \text{in } B_r(\bar{x}) \cap \Omega,$$

for some function ω_r which is nonnegative, continuous, stricly increasing in a neighbourhood of 0 *and satisfies* $\omega_r(0) = 0$.

Proof of case (a). Let $r = r(x_0)$ be defined in Lemma 4.4.7 for $\bar{x} = x_0$. We localize the argument in a ball of radius r around x_0 and we use the existence of the blow-up function U_r defined in Lemma 4.4.7 for $\bar{x} = x_0$. Let $\varepsilon > 0$. We double the variable and we consider (x,y) maximum point on $(\bar{B}_{\frac{r}{2}}(x_0) \cap \bar{\Omega}) \times (\bar{B}_{\frac{r}{2}}(x_0) \cap \bar{\Omega})$ of the function

$$\Phi(x,y) = u(x) - v(y) - \tilde{\phi}(x,y) \tag{4.107}$$

where

$$\tilde{\phi}(x,y) = \phi\left(\frac{(x+y)}{2}\right) + \frac{|x-y|^2}{\varepsilon^2} + k[U_r(x) + U_r(y)].$$

Note that, by (ii) of Lemma 4.4.7, we have that $(x,y) \in \bar{B}_{\frac{r}{2}}(x_0) \cap \Omega \times \bar{B}_{\frac{r}{2}}(x_0) \cap \Omega$; moreover, again by (ii) of Lemma 4.4.7, we have for k small enough

$$d(x), d(y) \geq \omega_r^{-1}\left(\frac{k}{2L}\right) =: \bar{\delta}, \tag{4.108}$$

where

$$L = ||u||_{L^\infty(\bar{B}_{\frac{r}{2}}(x_0)\cap\bar{\Omega})} + ||v||_{L^\infty(\bar{B}_{\frac{r}{2}}(x_0)\cap\bar{\Omega})} + ||\phi||_{L^\infty(\bar{B}_{\frac{r}{2}}(x_0)\cap\bar{\Omega})} + 1$$

Note that the existence of the blow-up function plays its mayor role here to get (4.108). This estimate tells us, roughly speaking, that the maximum points are away from the boundary. For fixed k, a standard argument shows that

$$\frac{|x-y|^2}{\varepsilon^2} \to 0 \text{ as } \varepsilon \to 0. \tag{4.109}$$

By the previous estimate on x,y and extracting subsequences if necessary, we can assume without loss of generality that when we send $\varepsilon, k \to 0$

$$x, y \to x_0, \quad u(x) - v(y) - \tilde{\phi}(x,y) \to u(x_0) - v(x_0) - \phi(x_0). \tag{4.110}$$

Let C_j be as in (J1) and $C_j^{-1}\frac{r}{4} > \xi' > 0$. We proceed as in Step 2 in the above proof and we write the viscosity inequalities

$$\begin{aligned} u(x) - v(y) &\leq H(y, -D[\tilde{\phi}(x,\cdot)](y)) - H(x, D[\tilde{\phi}(\cdot,y)](x)) \\ &+ \mathscr{I}^{\xi'}[u](x) - \mathscr{I}^{\xi'}[v](y) + \mathscr{I}_{\xi'}[\tilde{\phi}(\cdot,y)](x) - \mathscr{I}_{\xi'}[-\tilde{\phi}(x,\cdot)](y). \end{aligned} \tag{4.111}$$

Since $\tilde{\phi} \in C^1$ and by (J1), (M0), we have

$$\mathscr{I}_{\xi'}[\tilde{\phi}(\cdot,y)](x) \leq C_j ||D\tilde{\phi}||_{L^\infty(\bar{B}(x_0,\frac{r}{2}))} \int_{\mathbb{R}^n} 1_{|z|\leq\xi'}|z| d\mu_x(z) = o_{\xi'}(1), \tag{4.112}$$

where $o_{\xi'}(1)$ is independent of δ. The same holds for $-\mathscr{I}_{\xi'}[-\tilde{\phi}(\bar{x},\cdot)](\bar{y})$.

First we analyse the term $\mathscr{I}^{\xi'}[u](x)-\mathscr{I}^{\xi'}[v](y)$. For simplicity of exposition, we conclude the proof in the case the measure μ in the nonlocal terms has no dependence on x, i.e. $\mu_x\equiv\mu$. The result can be easily extended in the case of x-dependence analogously as already shown in Remark 4.4.5 for case (b) and (c). As done for case (b) and (c), we write

$$\mathscr{I}^{\xi'}[u](x)-\mathscr{I}^{\xi'}[v](y)=I^{\xi'}[J_x/J_y]+I^{\xi'}[J_y/J_x]+T^{\xi'}[J_x\cap J_y], \tag{4.113}$$

where we recall the following notations

$$J_x=\{z\in\mathbb{R}^n\,|\,x+j(x,z)\in\bar{\Omega}\} \tag{4.114}$$

and

$$I^{\xi'}[J_x/J_y]=\int_{\substack{J_x/J_y,\\ |z|\geq\xi'}} u(x+j(x,z))-u(x)d\mu(z); \tag{4.115}$$

$$I^{\xi'}[J_y/J_x]=\int_{\substack{J_y/J_x,\\ |z|\geq\xi'}} v(y)-v(y+j(y,z))d\mu(z);$$

$$T^{\xi'}[J_x\cap J_y]=\int_{\substack{J_x\cap J_y,\\ |z|\geq\xi'}}[u(x+j(x,z))-u(x)-(v(y+j(y,z))-v(y))]d\mu(z).$$

The estimation of the term $T^{\xi'}[J_x\cap J_y]$ is carried out as in the proof of Lemma 4.4.1. We give the details for completeness. Thanks to (4.110), we can take ε,k small enough so that $x,y\in B_{\frac{r}{4}}(x_0)\cap\Omega$. Since (x,y) is a maximum point of $u-v-\tilde{\phi}$ on $\bar{B}_{\frac{r}{2}}(x_0)\cap\bar{\Omega}\times\bar{B}_{\frac{r}{2}}(x_0)\cap\bar{\Omega}$, we have for $|z|\leq C_j^{-1}\frac{r}{4}$

$$u(x)-v(y)-\tilde{\phi}(x,y)\geq u(x+j(x,z))-v(y+j(y,z))-\tilde{\phi}(x+j(x,z),y+j(x,z)). \tag{4.116}$$

and then by the definition of $\tilde{\phi}$, we have

$$\begin{aligned} u(x+j(x,z))-u(x) \quad &- \quad (v(y+j(y,z))-v(y))\leq\frac{|x+j(x,z)-y-j(y,z)|^2}{\varepsilon^2}-\frac{|x-y|^2}{\varepsilon^2}\\ &+ \quad k[U_r(x+j(x,z))-U_r(x)]+k[U_r(y+j(y,z))-U_r(y)]\\ &+ \quad \phi((x+j(x,z)+y+j(y,z))/2)-\phi((x+y)/2). \end{aligned}$$

Note that by (J2) and (4.109), we have

$$\frac{|x+j(x,z)-y-j(y,z)|^2}{\varepsilon^2}-\frac{|x-y|^2}{\varepsilon^2}\leq|z|o_\varepsilon(1)$$

and then

$$\begin{aligned} u(x+j(x,z))-u(x) &- (v(y+j(y,z))-v(y)) \\ &\le |z|o_\varepsilon(1)+k[U_r(x+j(x,z))-U_r(x)]+k[U_r(y+j(y,z))-U_r(y)] \\ &+ \phi((x+j(x,z)+y+j(y,z))/2)-\phi((x+y)/2). \end{aligned} \tag{4.117}$$

Then for $0<\xi'<\xi<C_j^{-1}\frac{r}{4}$, by (4.117) and (M0), we get

$$\begin{aligned} T^{\xi'}[J_x\cap J_y] &\le k\mathscr{I}_\xi[U_r](x)+k\mathscr{I}_\xi[U_r](y)-k\mathscr{I}_{\xi'}[U_r](x)-k\mathscr{I}_{\xi'}[U_r](y) \\ &+ P_\xi-P_{\xi'}+K^\xi+o_\varepsilon(1), \end{aligned} \tag{4.118}$$

where $o_\varepsilon(1)$ is independent of ξ' and

$$K^\xi=\int_{\substack{J_x\cap J_y,\\ |z|\ge\xi}} u(x+j(x,z))-u(x)-(v(y+j(y,z))-v(y))d\mu(z),$$

$$P_\xi=\int_{\substack{J_x\cap J_y,\\ |z|\le\xi}} \phi((x+j(x,z)+y+j(y,z))/2)-\phi((x+y)/2)d\mu(z)$$

$$P_{\xi'}=\int_{\substack{J_x\cap J_y,\\ |z|\le\xi'}} \phi((x+j(x,z)+y+j(y,z))/2)-\phi((x+y)/2)d\mu(z)$$

Note that, since ϕ and U_r are lipschitz and by (J1) and (M0), we have

$$P_{\xi'}\le o_{\xi'}(1),\quad \mathscr{I}_{\xi'}[U_r](x)\le o_{\xi'}(1),\quad \mathscr{I}_{\xi'}[U_r](y)\le o_{\xi'}(1).$$

Then (4.118) becomes

$$T^{\xi'}[J_x\cap J_y]\le k\mathscr{I}_\xi[U_r](x)+k\mathscr{I}_\xi[U_r](y)+P_\xi+K^\xi+o_{\xi'}(1)+o_\varepsilon(1), \tag{4.119}$$

where $o_\varepsilon(1)$ is independent of ξ'.

Now we estimate the left terms $I^{\xi'}[J_x/J_y]$ and $I^{\xi'}[J_y/J_x]$ in (4.113). Thanks to the estimate (4.108) on x,y, in this case the estimation is easier than in the previous cases (b) and (c). Take for example $I^{\xi'}[J_x/J_y]$ (the argument being analogous for $I^{\xi'}[J_y/J_x]$) and note that by (4.108) the integral is independent of ξ' as soon as $\xi'<\bar\delta$ where $\bar\delta$ is defined in (4.108). Then by the boundedness of u, we have

$$I^{\xi'}[J_x/J_y]\le 2C||u||_\infty\int_{|z|\ge\bar\delta} 1_{J_x/J_y}d\mu(z)$$

and since $|J_x/J_y|\to 0$ as $\varepsilon\to 0$, by (M0) and the Dominated Convergence theoremm we get

$$I^{\xi'}[J_x/J_y]\le o_\varepsilon(1), \tag{4.120}$$

where $o_\varepsilon(1)$ is independent of ξ'. Then plugging (4.120) and (4.119) into (4.113) and then coupling it with (4.111) and (4.112), we get for $C_j^{-1}\frac{r}{4} > \xi' > 0$

$$\begin{aligned} u(x) - v(y) &\leq H(y, -D[\tilde{\phi}(x,\cdot)](y)) - H(x, D[\tilde{\phi}(\cdot,y)](x) \\ &+ k\mathscr{I}_\xi[U_r](x) + k\mathscr{I}_\xi[U_r](y) + P_\xi + K^\xi + o_\varepsilon(1) + o_{\xi'}(1), \end{aligned} \tag{4.121}$$

where $o_\varepsilon(1)$ is independent of ξ'. Moreover, note that by (i) of Lemma 4.4.7 we have

$$\begin{aligned} H(y, -D[\tilde{\phi}(x,\cdot)](y)) - H(x, D[\tilde{\phi}(\cdot,y)](x) + k\mathscr{I}_\xi[U_r](x) + k\mathscr{I}_\xi[U_r](y) \\ \leq B|D\phi((x+y)/2)| + o_\varepsilon(1). \end{aligned} \tag{4.122}$$

Indeed, by (i) of Lemma 4.4.7, we estimate the integrals terms of the left hand side of (4.122) together with the first order terms involving U_r in $H(y, -D[\tilde{\phi}(x,\cdot)](y)) - H(x, D[\tilde{\phi}(\cdot,y)](x)$. The remaining terms in the Hamiltonians are treated analogously as already showed in the proof of $b)$ and $c)$. Then, plugging (4.122) into (4.121), we get

$$u(x) - v(y) \leq B|D\phi((x+y)/2)| + P_\xi + K^\xi + o_\varepsilon(1) + o_{\xi'}(1), \tag{4.123}$$

where $o_\varepsilon(1)$ is independent of ξ'. The rest of the proof is the same as in the previous cases, by sending first $\xi' \to 0$, then $\varepsilon \to 0$. For the details we refer to the end of the proof above. □

Now we prove Theorem 4.2.6 for H of Bellman type.

Proof of Theorem 4.2.6. By contradiction, we suppose that

$$M = \sup_\Omega \{u - v\} > 0. \tag{4.124}$$

Denote $\omega(x) = u(x) - v(x)$ and for $0 < \nu$, consider

$$\Phi(x) = \omega(x) - \psi(R^{-1}|x|) + \nu d(x)$$

where ψ is a smooth function such that

$$\psi(s) = \begin{cases} 0 & \text{for } 0 \leq s < \frac{1}{2}, \\ \text{increasing} & \text{for } \frac{1}{2} \leq s < 1, \\ ||u||_\infty + ||v||_\infty + 1 & \text{for } s \geq 1. \end{cases} \tag{4.125}$$

and d is the signed distance from the boundary (in a suitable neighbourhood of the boundary and bounded in all the domain, see Remark 4.2.1). Note that $\sup\Phi \to M$ as $R \to \infty$ and $\nu \to 0$. Since $\Phi \leq -1/2$ for $|x|$ large and ν small and $M > 0$, the function Φ achieves its positive maximum $\sup\Phi > \frac{M}{2}$ at a point x for R big and ν small enough. We give the details in the case where all maximum points x are located on the boundary. We have

$$\omega(x) = M + o_{R,\nu}(1) \tag{4.126}$$

where with $o_{R,\nu}(1)$ we mean that the limit is zero if $R \to \infty, \nu \to 0$. We use $\phi(\cdot) := \psi(R^{-1}|\cdot|) - \nu d(\cdot)$ as a test function at x. We suppose $x \in \partial\Omega$ and we observe that

$$\frac{\partial \phi}{\partial n} \geq -R^{-1}||\psi'||_{L^\infty} + \nu > 0, \tag{4.127}$$

where ψ is defined in (4.125) and the last inequality holds for $\nu > R^{-1}||\psi'||_{L^\infty}$.
By Lemma 4.4.1, ω is a viscosity subsolution of

$$\begin{cases} \omega - \mathscr{I}[\omega](x) - B|D\omega| \leq 0 & \text{in } \Omega \\ \frac{\partial \omega}{\partial n} = 0 & \text{on } \partial\Omega \end{cases}$$

and then by (4.127), we get

$$\omega(x) - \mathscr{I}[\phi](x) - B(|D\phi(x)|) \leq 0 \ \text{ in } \ \Omega.$$

By Lemma 5.3.10 (see Appendix C), we have

$$\mathscr{I}[\phi](\cdot) \leq o_{\nu,R}(1), \quad |D\phi(\cdot)| \leq o_{\nu,R}(1)$$

and by (4.126), we get

$$M + o_{\nu,R}(1) \leq o_{\nu,R}(1)$$

and by letting $R \to \infty$, $\nu \to 0$, we get a contradiction since $M > 0$ and we conclude the proof.
□

4.4.2 Coercive Hamiltonians

We recall that we denote by coercive Hamiltonian an Hamiltonian satisfying (H1) which can be either of sublinear type satisfying (Ha) or superlinear type satisfying (Hb) and (Hc). We proceed analogously as for Hamiltonian of Bellman type and we prove Lemma 4.4.8. Once proved Lemma 4.4.8, the proof of Theorem 4.2.6 for H coercive follows by standard arguments as already showed for Hamiltonian of Bellman type. We sketch first the proof of Theorem 4.2.6 and then we prove Lemma 4.4.8.

Proof of Theorem 4.2.6. We just observe that we proceed again by contradiction, supposing that

$$M = \sup_{\Omega}\{u - v\} > 0. \tag{4.128}$$

We fix $0 < \mu < 1$ and define

$$\omega_\mu(x) = \mu u(x) - v(x) \quad x \in \Omega.$$

We proceed as in the Bellman case and we use Lemma 4.4.8 to get

$$M + o_{\nu,R,\mu}(1) - o_{\nu,R}(1) \leq CA(1-\mu).$$

Then, by letting $R \to \infty$, $\nu \to 0$ and finally $\mu \to 1$, we get a contradiction since $M > 0$ and we conclude the proof. □

Lemma 4.4.8. *Let* $\mathscr{I}$ *as in* (4.5) *and assume* μ *satisfies* $(M0),(M1),(M2)$*,* j *satisfies* $(J0),(J1),(J2)$*. Let* H *be a coercive Hamiltonian and let* u,v *be respectively bounded sub and supersolutions to* (4.4)*. Let* $\mu \in (0,1)$ *if* H *is superlinearly coercive,* $\mu = 1$ *is* H *is sublinearly coercive. Then the function*

$$\omega(x) := \mu u(x) - v(x)$$

satisfies, in the viscosity sense, the equation

$$\begin{cases} \omega - \mathscr{I}[\omega](x) - C_{m,\mu}|D\omega|^m \leq A(1-\mu) & \text{in } \Omega \\ \frac{\partial \omega}{\partial n} = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.129}$$

where $A, C_{m,\mu}$ *are positive constants which depend on the data. Precisely, if* $\mu = 1$*,* $C_{m,\mu} = \tilde{C}$ *where* $\tilde{C}$ *is defined in (Hb) and, if* $\mu \in (0,1)$*,* $C_{m,\mu} = \bar{C}^{1-m}C^m 2^{m-1} m^{-m}(1-\mu)^{1-m}$

Proof. We give the details when H has superlinear form i.e. when $m > 1 > \sigma$, since the proof in the sublinear case is similar with easier computations. Since the proof is similar to that of Lemma 4.4.1, we focus only on the main differences.

We start by noting that if u is a subsolution of (4.4), then

$$\bar{u} = \mu u$$

is a viscosity subsolution to

$$\bar{u} - \mathscr{I}[\bar{u}](x) + \mu H(x, \mu^{-1} D\bar{u}) \leq 0, \quad \text{in } \Omega. \tag{4.130}$$

Let $x_0 \in \bar{\Omega}$ and ϕ a smooth function such that $\omega - \phi$ has a strict maximum point. We suppose that $x_0 \in \partial\Omega$, the other case being similar and even simpler.

Step. *1-Localising on equidistant points (i.e.* $d(x) = d(y)$*)* We double the variable and we consider the function

$$\Phi(x,y) := \bar{u}(x) - v(y) - \tilde{\phi}(x,y) \tag{4.131}$$

where $\tilde{\phi}$ is as in (4.45) with $K = 2$. Let C_j be defined in (J1). We prove that the maximum point (x,y) of Φ over the set $A := \left(\bar{B}_{2C_j}(x_0) \cap \bar{\Omega}\right) \times \left(\bar{B}_{2C_j}(x_0) \cap \bar{\Omega}\right)$ satisfies as $\varepsilon \to 0$

$$x, y \to x_0, \quad \varepsilon^{-1}\chi_\varepsilon(|x-y|) \to 0, \quad \varepsilon^{-1}|d(x) - d(y)| \to 0 \tag{4.132}$$

and

$$\bar{u}(x) - v(y) - \tilde{\phi}(x,y) \to \bar{u}(x_0) - v(x_0) - \phi(x_0). \tag{4.133}$$

Moreover, we suppose that

$$\frac{\partial \phi}{\partial n}(x_0) > 0, \tag{4.134}$$

then for ε small enough

$$\frac{\partial \phi}{\partial n}((x+y)/2) > \frac{1}{2}\frac{\partial \phi}{\partial n}(x_0) > 0. \tag{4.135}$$

By contradiction, we suppose that $d(x) > d(y)$. By (4.135), we have for $0 < \xi' < 2$ and $0 < \mu < 1$

$$\mu u(x) - \mu \mathscr{I}^{\xi'}[u](x) - \mathscr{I}_{\xi'}[\tilde{\phi}(\cdot,y)](x) + \mu H\left(x, \mu^{-1}D[\tilde{\phi}(\cdot,y)](x)\right) \leq 0. \tag{4.136}$$

Note that

$$|D[\tilde{\phi}(\cdot,y)](x)| \geq \varepsilon^{-1} - C, \tag{4.137}$$

where $C > 0$ is a constant independent of ε. For the integral terms in (4.136) we proceed as in Lemma 4.4.1 by using Lemma 4.4.3. For the Hamiltonian terms we use assumption (H1) toghether with (4.137), and by the boundedness of u, we write (4.136) as follows

$$\varepsilon^{-m}\left(-\varepsilon^{m-\sigma} + c_0\mu^{1-m}\right) \leq C,$$

and we get a contradiction for ε small, since $\sigma < m$.

Step. *2-Writing the viscosity inequalities and sending the parameters to their limits* We regularize the test function $\tilde{\phi}$ as in (4.63) and we denote

$$\Phi(x,y) = \bar{u}(x) - v(y) - \tilde{\phi}(x,y),$$

and by $(\bar{x},\bar{y})$ the maximum point of Φ. We have

$$d(\bar{x}) \to d(\bar{y}), \quad u(\bar{x}) \to u(x), \quad v(\bar{y}) \to v(y) \text{ as } \delta \to 0. \tag{4.138}$$

where (x,y) is a maximum point of Φ defined in (4.131). We write the viscosity inequalities for u and v for any $0 < \xi' < 1$

$$\begin{aligned} u(\bar{x}) - v(\bar{y}) \quad \leq \quad & \mu H(\bar{y}, \mu^{-1}D_x\tilde{\phi}(\cdot,\bar{y})(\bar{x})) - H(\bar{x}, -D_y\tilde{\phi}(\bar{x},\cdot)(\bar{y})) \\ + \quad & \mathscr{I}^{\xi'}[u](\bar{x}) - \mathscr{I}^{\xi'}[v](\bar{y}) + \mathscr{I}_{\xi'}[-\tilde{\phi}(\cdot,\bar{y})](\bar{x}) - \mathscr{I}_{\xi'}[-\tilde{\phi}(\bar{x},\cdot)](\bar{y}), \end{aligned}$$

and since

$$\mathscr{I}_{\xi'}[-\tilde{\phi}(\cdot,\bar{y})](\bar{x}) \le o_{\xi'}(1), \quad -\mathscr{I}_{\xi'}[-\tilde{\phi}(\bar{x},\cdot)](\bar{y}) \le o_{\xi'}(1),$$

where $o_{\xi'}(1)$ is independent of δ, we have

$$\begin{aligned} u(\bar{x}) - v(\bar{y}) \le \mu H(\bar{y}, \mu^{-1} D_x \tilde{\phi}(\cdot,\bar{y})(\bar{x})) \quad &- \quad H(\bar{x}, -D_y \tilde{\phi}(\bar{x},\cdot)(\bar{y})) \\ &+ \quad \mathscr{I}^{\xi'}[u](\bar{x}) - \mathscr{I}^{\xi'}[v](\bar{y}) + o_{\xi'}(1). \end{aligned} \tag{4.139}$$

Denote

$$\mathscr{H} = \mu H(\bar{y}, \mu^{-1} D[\tilde{\phi}(\cdot,\bar{y})](\bar{x})) - H(\bar{x}, -D[\tilde{\phi}(\bar{x},\cdot)](\bar{y})). \tag{4.140}$$

We recall that

$$D[\tilde{\phi}(\cdot,\bar{y})](\bar{x}) = \varepsilon^{-1}[\chi'_\varepsilon(|\bar{x}-\bar{y}|)\hat{p} - 2\chi'_\delta(|d(\bar{x})-d(\bar{y})|)\tilde{p}n(\bar{x})] + D\phi((x+y)/2)/2, \tag{4.141}$$

$$D[-\tilde{\phi}(\bar{x},\cdot)](\bar{y}) = \varepsilon^{-1}[\chi'_\varepsilon(|\bar{x}-\bar{y}|)\hat{p} - 2\chi'_\delta(|d(\bar{x})-d(\bar{y})|)\tilde{p}n(\bar{y})] - D\phi((x+y)/2)/2 \tag{4.142}$$

where

$$\hat{p} = \frac{\bar{x}-\bar{y}}{|\bar{x}-\bar{y}|}, \quad \tilde{p} = \frac{d(\bar{x})-d(\bar{y})}{|d(\bar{x})-d(\bar{y})|}. \tag{4.143}$$

Note that

$$|D[\tilde{\phi}(\cdot,\bar{y})](\bar{x}) - D[-\tilde{\phi}(\bar{x},\cdot)](\bar{y})| \le 4\varepsilon^{-1}|\bar{x}-\bar{y}| + |D\phi((\bar{x}+\bar{y})/2)|$$

and

$$|D[\tilde{\phi}(\cdot,\bar{y})](\bar{x})|, |-D[\tilde{\phi}(\bar{x},\cdot)](\bar{y})| \le q$$

where

$$q = 3\varepsilon^{-1} + 2^{-1}||D\phi||_{L^\infty(B_{2C_j}(x_0))}.$$

Thanks to (Hb) and (Hc), we get for ε small enough

$$\begin{aligned} \mathscr{H} \ge (1-\mu)(m-1)\bar{C}q^m - A(1-\mu) - \omega_1(|\bar{x}-\bar{y}|)(1+q^m) \\ - C|D\phi((\bar{x}+\bar{y})/2)|q^{m-1} - 4C\varepsilon^{-1}|\bar{x}-\bar{y}|q^m \end{aligned}$$

Note that, by (4.132) and (4.138), we can take $\varepsilon = \varepsilon(\mu), \delta = \delta(\mu)$ small enough so that

$$(1-\mu)(m-1)\bar{C} - \omega_1(|\bar{x}-\bar{y}|) - 4C\varepsilon^{-1}|\bar{x}-\bar{y}| > 0$$

and we can write

$$\begin{aligned} \mathscr{H} \quad &\ge \quad (1-\mu)(m-1)\bar{C}q^m/2 - C|D\phi((\bar{x}+\bar{y})/2)|q^{m-1} - A(1-\mu) - o_{\delta,\varepsilon}(1) \\ &\ge \quad \inf_{q\ge 0}\{(1-\mu)(m-1)\bar{C}q^m/2 - C(|D\phi((\bar{x}+\bar{y})/2)|)q^{m-1}\} - B(1-\mu) - o_{\delta,\varepsilon}(1), \end{aligned}$$

where $o_{\delta,\varepsilon}(1)$ means that $\lim_{\delta\to 0} o_{\delta,\varepsilon}(1) = o_{\varepsilon}(1)$. Note that the infimum in the previous expression is attained and therefore

$$\mathscr{H} \geq -C_{m,\mu}|D\phi((\bar{x}+\bar{y})/2)|^m - A(1-\mu) - o_{\delta,\varepsilon}(1), \tag{4.144}$$

where $C_{m,\mu} = \bar{C}^{1-m}C^m 2^{m-1}m^{-m}(1-\mu)^{1-m}$. Then we couple (4.144), (4.140), (4.139) and as in Lemma 4.4.1 we let $\delta \to 0$, getting

$$\begin{aligned} u(x) - v(y) \quad &- \quad C_m(1-\mu)^{1-m}|D\phi((x+y)/2)|^m - A(1-\mu) - o_{\varepsilon}(1) \\ &\leq \quad \mathscr{I}^{\xi'}[u](x) - \mathscr{I}^{\xi'}[v](y) + o_{\xi'}(1). \end{aligned} \tag{4.145}$$

We estimate the nonlocal terms in (4.145) as in Lemma 4.4.1 (steps 2, 3) getting for $\xi' < \xi < 1$

$$\mathscr{I}^{\xi'}[u](x) - \mathscr{I}^{\xi'}[v](y) \leq P_{\xi} + K^{\xi} + o_{\varepsilon}(1) + o_{\xi'}(1). \tag{4.146}$$

and note that $o_{\varepsilon}(1)$ is independent of ξ'. Then we plug (4.146) into (4.145) and we let $\xi' \to 0$, getting

$$\mu u(x) - v(y) - C_m(1-\mu)^{1-m}|D\phi(2^{-1}(x+y))|^m \leq P_{\xi} + K^{\xi} + A(1-\mu) + o_{\varepsilon}(1) \tag{4.147}$$

and we conclude sending $\varepsilon \to 0$ and using (4.101) and (4.102) as in Lemma 4.4.1.

$\square$

Chapter 5

Applications to evolutive problems: existence, uniqueness and asymptotic behavior

In this chapter we present some applications of the results proved in Chapter 4 in the stationary case to the evolutive setting and we consider the associated Cauchy problem

$$\begin{cases} \partial_t u - \mathscr{I}[u(\cdot,t)](x) + H(x,t,u,Du) = 0 & \text{in } \Omega \times (0,+\infty) \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega \times (0,\infty), \\ u(x,0) = u_0(x) & \text{in } \bar{\Omega}, \end{cases} \tag{5.1}$$

where $\Omega \subset \mathbb{R}^n$ is smooth enough satisfying assumption (O) (i.e. of class $W^{2,\infty}$), $u_0 \in C(\Omega)$, $\mathscr{I}[u]$ is an integro-partial differential operator of censored type and of order striclty less than 1, defined as

$$\mathscr{I}[u(\cdot,t)](x) = \lim_{\delta \to 0^+} \int_{\substack{|z| > \delta, \\ x + j(x,z) \in \bar{\Omega}}} [u(x + j(x,z),t) - u(x,t)] d\mu_x(z), \tag{5.2}$$

where μ_x is a singular nonnegative Radon measure (depending on x under some suitable assumptions) satisfying (M0), (M1) (see Chapter 4, Section 4.2), and $j(x,z)$ is a jump function satisfying (J0), (J1), (J2) (see Chapter 4, Section 4.2). The main example are measures μ_x with density $\frac{d\mu_x}{dz} = g(x,z)|z|^{-(n+\sigma)}$ with $\sigma < 1$ and g a nonnegative bounded function Lipschitz in x, uniformly with respect to z.

Note that the operator in (5.2) is the natural extension to the evolutive case of the nonlocal operator considered in the stationary case and defined in (4.5), Chapter 4. Moreover H : $\bar{\Omega} \times [0,+\infty) \times \mathbb{R} \times \mathbb{R}^n \mapsto \mathbb{R}$ is a continuous function whose growth in the gradient makes it the leading order term in the equation, and can be coercive or of Bellman type. We refer to the following section for the precise assumptions on the Hamiltonian. The well-posedness of problems as (5.1) follows from analogous arguments used for the stationary problem with some standard adaptations. We refer to Definition (5.1.1) for the definition of solutions and to Theorem 5.2.1 and Theorem 5.2.4 for further details and proofs of uniqueness and existence.

We study two different kind of asymptotic behaviour of the solution of (5.1). First we consider an Hamiltonian either in coercive form or of Bellman type under assumptions ensuring uniqueness of the solution of the associated stationary problem. We prove, by classical methods based on the weak-relaxed semilimits, the convergence as $t \to +\infty$ of the solution of (5.1) to the unique solution of the associated stationary problem.

On the other hand, when the associated stationary problem has not unique solution, we consider an Hamiltonian with superfractional coercive growth and we study the so-called *ergodic large time behaviour*, proving that the solution of (5.1) approaches a solution of the so-called *ergodic problem* as $t \to +\infty$. We follow the methods of [29]. In particular, we rely on the Hölder regularity up to the boundary and a control of the oscillation of subsolutions, which has been proved by Barles, Ley Koike, Topp in [29] (see also Barles and Topp [33]), in the case of censored operators and coercive Hamiltonian with $m > 1$.

5.1 Assumptions and definitions of solutions

We are going to consider the finite time horizon problem associated to (5.1)

$$\begin{cases} \partial_t u - \mathscr{I}[u(\cdot,t)](x) + H(x,t,u,Du) = 0 & \text{in } \Omega \times (0,T] \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u_0(t) & \text{in } \bar{\Omega}. \end{cases} \tag{5.3}$$

We remark that the definition of viscosity solutions to (5.3) (and then to (5.1)) is the natural extension of Definition 4.2.5 to the corresponding Cauchy problem. For convenience of the reader, we give it in the following definition Definition 5.1.1.

We denote

$$F(x,t,u,\phi) = \partial_t \phi(x,t) - \mathscr{I}_\xi[\phi(\cdot,t)](x) - \mathscr{I}^\xi[u(\cdot,t)](x) + H(x,t,u(x,t),D\phi(x,t)).$$

Definition 5.1.1. *(i) A function $u \in BUSC(\bar{\Omega} \times [0,T])$ is a viscosity subsolution to* (5.3) *if, for any test-function $\phi \in C^1(\mathbb{R}^N \times [0,T])$ and maximum point (x,t) of $u - \phi$ in $\bar{B}_{C_j\xi}(x) \times (t-\xi, t+\xi) \cap \bar{\Omega} \times [0,T]$, we have the inequality*

$$\begin{aligned} F(x,t,u,\phi) &\le 0 \qquad if \quad (x,t) \in \Omega \times (0,T] \\ \min\{F(x,t,u,\phi), \frac{\partial \phi}{\partial n}(x,t)\} &\le 0 \qquad if \quad x \in \partial\Omega \\ \min\{F(x,t,u,\phi), u(x,t) - u_0(t)\} &\le 0 \qquad if \quad t = 0. \end{aligned}$$

(ii) A bounded lsc function u is a viscosity supersolution to (5.3) *if, for any test-function $\phi \in C^1(\mathbb{R}^n \times [0,T])$ and minimum point (x,t) of $u - \phi$ in $\bar{B}_{C_j\xi}(x) \times (t-\xi, t+\xi) \cap \bar{\Omega} \times [0,T]$,*

we have the inequality

$$
\begin{aligned}
F(x,t,u,\phi) \geq 0 \qquad & if \quad (x,t) \in \Omega \times (0,T] \\
\max\{F(x,t,u,\phi), \frac{\partial \phi}{\partial n}(x,t)\} \geq 0 \qquad & if \quad x \in \partial\Omega \\
\max\{F(x,t,u,\phi), u(x,t) - u_0(t)\} \geq 0 \qquad & if \quad t = 0.
\end{aligned}
$$

(iii) A viscosity solution is both a sub- and supersolution.

Remark 5.1.2. The Definition 5.1.1 interprets the point at $\Omega \times \{T\}$ as *interior points*, which is consistent with the classical definition of the Cauchy problem for parabolic equations (see [77],[91]). We can obtain the same results by using a weaker definition of viscosity solution (concerning functions defined only in $\bar{\Omega} \times [0,T)$). Since this concern wouldn't bring any important contribution to the development of our results, we avoid this extra difficulty.

We also assume the following condition. We remark that, in order to prove the comparison principle (Theorem 5.2.1), (H') is assumed mainly for simplicity of exposition and can be relaxed as observed in the Remark 5.1.3.

(H') For all $R > 0$, there exists $\gamma_R \geq 0$ such that for all $x \in \bar{\Omega}, u, v \in \mathbb{R}, |u|, |v| \leq R, 0 \leq t \leq R$ and $p \in \mathbb{R}^n$, we have

$$H(x,t,u,p) - H(x,t,v,p) \geq \gamma_R(u - v).$$

Remark 5.1.3. In order to prove the comparison principle stated in Theorem 5.2.1, assumption (H') can be relaxed assuming only $\gamma_R \in \mathbb{R}$ (i.e. without requiring the non negativity). Indeed, in the proof of the comparison principle (see Lemma 5.2.3 and Theorem 5.2.1) we will take γ_R as in (H') for $R = ||u||_\infty + ||v||_\infty$. In this case note that, without loss of generality, we can suppose $\gamma_R \geq 1$. Indeed if $\gamma_R < 1$ we perform the change $\tilde{u} = ue^{-(\gamma_R - 1)t}$ (analogously for the supersolution) and prove Lemma 5.2.3 and Theorem 5.2.1 for $\tilde{u}$ and $\tilde{v}$.

As it is classical in viscosity solutions theory, the comparison principle allows the application of Perron's method to conclude the existence. To this end, we introduce the following assumption, which will allows us to build sub and supersolutions:

(E') For all $T > 0, R > 0$ there exists a constant $H_R > 0$ such that

$$||H(x,t,r,p)||_\infty \leq H_R \;\forall x \in \Omega, t \in [0,T], r, p \in \mathbb{R}, |r|, |p| \leq R.$$

Hamiltonian in Bellman form

We say that the Hamiltonian H has a Bellman form if for $t \in [0,+\infty), x \in \bar{\Omega}, p \in \mathbb{R}^n, H(x,t,r,p)$ can be written as

$$H(x,t,r,p) = \sup_{\alpha \in \mathscr{A}} \{\lambda(x,t,\alpha)r - b(x,t,\alpha) \cdot p - l(x,t,\alpha)\}; \tag{5.4}$$

where $b, \lambda : \bar{\Omega} \times [0,+\infty) \times \mathscr{A} \to \mathbb{R}^n$ and $l : \bar{\Omega} \times [0,+\infty) \times \mathscr{A} \to \mathbb{R}$, are continuous and bounded functions and satisfy the following properties.

(C') *Uniform continuity of l and λ*:
There exist modulus of continuity ω_l, ω_λ such that such that $\forall \alpha \in \mathscr{A}, \forall x, y \in \bar{\Omega}, t, s \in [0,+\infty)$

$$|l(x,t,\alpha) - l(y,s,\alpha)| \leq \omega_l(|x-y| + |t-s|);$$

$$|\lambda(x,t,\alpha) - \lambda(y,s,\alpha)| \leq \omega_\lambda(|x-y| + |t-s|);$$

(L') *Uniform Lipschitz continuity of the drift b*: There exists $C > 0$ such that $\forall \alpha \in \mathscr{A}\, \forall (x,s),(y,t) \in \bar{\Omega} \times [0+\infty)$

$$|b(x,s,\alpha) - b(y,t,\alpha)| \leq C(|x-y| + |s-t|)$$

Remark 5.1.4. Note that assumption (L') may seem quite unusual since it requires also the uniform Lipschitz continuity of the drift b in the time variable. This is due to the fact that the proof of Lemma 4.4.1 (and Lemma 4.4.8), in particular in step 1, relies on an asymmetric use of the viscosity inequalities satisfied by the sub- and supersolution. For this reason the standard approach by the Ishi's Lemma for evolutive equation is not applicable and we need to double also the time variable in the test function, which as a consequence will have the same dependence on x and t. For a full explanation of the need of this assumption, we refer to the proof of Lemma 5.2.3.

We also introduce the notations

$$\Gamma_{\text{in}} = \{(x,t) \in \partial\Omega \times (0,+\infty) : b(x,t,\alpha) \cdot n(x) < 0\, \forall \alpha \in \mathscr{A}\};$$

$$\Gamma_{\text{out}} = \{(x,t) \in \partial\Omega \times (0,+\infty) : b(x,t,\alpha) \cdot n(x) > 0\, \forall \alpha \in \mathscr{A}\};$$

$$\Gamma := \{(x,t) \in \partial\Omega \times (0,+\infty) \,|\, \exists \alpha_1, \alpha_2 \in \mathscr{A} \text{ s. t. } b(x,t,\alpha_1) \cdot n(x) < 0, b(x,t,\alpha_2) \cdot n(x) > 0\}.$$

In order to avoid two completely different drift's behavior for abitrarily closed points, we assume that each of these subsets is uniformly away from the others, as encoded in the following assumptions (B1') and (B2'). For example, if $\partial\Omega \times (o,+\infty)$ is connected, then it consists in one piece belonging to one of Γ_{in}, Γ_{out} and Γ; otherwise, we are able to deal with boundary with several components of different types, precisely each one belonging to one between Γ_{in}, Γ_{out} and Γ.

The assumptions we do on these subsets are the following

$$\Gamma_{\text{in}} \cup \Gamma_{\text{out}} \cup \Gamma = \partial\Omega \times (0,+\infty) \tag{B1'}$$

and

$$\Gamma_{\text{in}}, \Gamma_{\text{out}}, \Gamma \text{ are unions of connected components of } \partial\Omega \times (0,+\infty). \tag{B2'}$$

Remark 5.1.5. The same remark as in the stationary case holds. In particular, note that the strict positivity of the drift in the previous assumptions is fundamental, since it allows us to control the growth of the nonlocal terms (wich we recall are of order strictly less than 1) by the gradient in the Hamiltonian.

We remark also that in order to treat the points of Γ_{in}, we use the existence of a blow-up supersolution exploding on the boundary. Here the strict positivity of the drift term on the points of Γ_{in} is essential to control the growth on the boundary of the integral term computed on this blow-up supersolution. We refer to the proof of Lemma 4.4.1 and in particular to Lemma 4.4.7 for further details.

Coercive Hamiltonian

In the case of coercive Hamiltonians, we restrict the time dependence of H by the assumption:

(H0') There exist $H_0 : \bar{\Omega} \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ continuous and $f : \bar{\Omega} \times [0,\infty) \to \mathbb{R}$ uniformly continuous and bounded such that for all $x \in \bar{\Omega}, r \in \mathbb{R}, p \in \mathbb{R}^n$

$$H(x,t,r,p) = H_0(x,r,p) - f(x,t),$$

We consider *superfractional* coercive Hamiltonians, where the gradient growth is given by H_0 through the assumption:

(H1') For all $R > 0$, there exist $m > \sigma, c_0 > 0, A_R > 0$ such that for all $x \in \bar{\Omega}, p \in \mathbb{R}^n, |r| \leq R$

$$H_0(x,r,p) \geq c_0 |p|^m - A_R.$$

We split the analysis depending on the gradient growth of H_0:

Sublinear coercivity: Assume (H0') holds. We say that H is *sublinear coercive* if H_0 satisfies (H1') with $m \leq 1$ and the following continuity condition holds :

(Ha') For all R there exist a constant $\tilde{C}_R > 0$ and a modulus of continuity ω_R such that for all $x, y \in \bar{\Omega}, p, q \in \mathbb{R}^n, |r| \leq R$

$$H_0(y,r,p) - H_0(x,r,q) \leq \omega_R(|x-y| + |x-y||p|) + \tilde{C}_R |p-q|.$$

Superlinear coercivity: Assume (H0') holds. We say that H is *superlinearly* coercive if H_0 satisfies:

(Hb') There exist $m > 1, \bar{C} > 0$, such that for all $R > 0$ there exists $A_R > 0$ such that

$$H_0(x,r,p) - \mu H_0(x,\mu^{-1}r,\mu^{-1}p) \leq (1-\mu)\left(\bar{C}(1-m)|p|^m + A_R\right),$$

for all $\mu < 1, x \in \bar{\Omega}, p \in \mathbb{R}^n, |r| \leq R$;

(Hc') If m is given by assumption (Hb'), for all $R > 0$ there exist a constant $C_R > 0$ and a modulus of continuity ω_R such that

$$H_0(y,r,p) - H_0(x,r,q) \leq \omega_R(|x-y|)(1+|p|^m \vee |q|^m) + C_R|p-q|(|p|^{m-1} \vee |q|^{m-1}),$$

for all $x,y \in \bar{\Omega}, |r| \leq R, q,p \in \mathbb{R}^n$.

5.2 The comparison principle

We recall that we denote by Hamiltonian of Bellman type an Hamiltonian defined as in (5.4) satisfying (C'),(L'), (B1'), (B2') and by coercive Hamiltonian an Hamiltonian satisfying (H0'), (H1') which can be either of sublinear type satisfying (Ha') or superlinear type satisfying (Hb') and (Hc'). We prove the following comparison principle for the problem (5.1).

Theorem 5.2.1. *Let Ω be an open subset of $\mathbb{R}^n$ satisfying (O), $u_0 \in C(\bar{\Omega})$. Assume $(M0), (M1)$, $(J0)$, $(J1)$, $(J2)$ and let H be an Hamiltonian either of Bellman type or a coercive Hamiltonian satisfying (H'). Let $u, v \in L^\infty(\bar{\Omega} \times [0,T])$ for all $T > 0$ be respectively a usc sub and lsc supersolution of* (5.1). *Then*

$$u \leq v \quad \text{in } \bar{\Omega} \times [0,+\infty).$$

Before proving the theorem, we remark that, as considered as a part of the parabolic boundary, we ask the initial condition to be satisfied in the generalized sense. However, the initial condition is satisfied in the classical sense on $\Omega \times \{0\}$, as we prove in the following lemma.

Lemma 5.2.2. *Assume Ω is a open subset of $\mathbb{R}^n$ and $H \in C(\bar{\Omega} \times [0,+\infty) \times \mathbb{R} \times \mathbb{R}^n), u_0 \in C(\bar{\Omega})$. If u, v are respectively a BUSC viscosity subsolution and a BLSC viscosity supersolution to* (5.1)*, then $u(x,0) \leq u_0(x) \leq v(x,0)$ for all $x \in \bar{\Omega}$.*

Proof. The proof is similar to that of analogous result for second-order case and Dirichlet boundary conditions presented in [68], with some modifications due to the presence of the nonlocal operator, to Neumann conditions and to the unboundedness of the domain. We give the details for completeness. We assume by contradiction that

$$u(x_0,0) > u_0(x_0) \tag{5.5}$$

for some $x_0 \in \bar{\Omega}$. Suppose that $x_0 \in \partial\Omega$, the other case being simpler. For all $\varepsilon > 0, \alpha > 0$ such that $\alpha \ll \varepsilon$, and $\nu > 0, R > 0$, we consider

$$\Phi(x,t) = u(x,t) - \phi(x,t) - u_0(x_0),$$

where

$$\phi(x,t) = e^{-Kd(x)} \frac{|x_0 - x|^2}{\varepsilon^2} + \frac{t}{\alpha} - \nu d(x) + \psi\left(\frac{|x - x_0|}{R}\right), \tag{5.6}$$

where d is the signed distance from the boundary (in a suitable neighbourhood of the boundary and bounded in all the domain, see Remark 4.2.1) and K in (5.6) is a positive constant such that $K > ||D^2 d||_\infty + 1$ (in the neighbourhood of the boundary where d is smooth) and

$$\psi(s) = \begin{cases} 0 & \text{for } 0 \leq s < \frac{1}{2}, \\ \text{increasing} & \text{for } \frac{1}{2} \leq s < 1, \\ 2||u||_\infty + 1 & \text{for } s \geq 1. \end{cases} \tag{5.7}$$

We observe that

$$\Phi(x_0, 0) = u(x_0, 0) - u_0(x_0) > 0,$$

and by the definition of ψ, we have for $|x - x_0|$ large enough and ν small enough

$$\Phi(x,t) < 0.$$

Then, we have that $M := \max_{\bar{\Omega}\times[0,T]} \Phi = \Phi(\bar{x},\bar{t})$ for some $(\bar{x},\bar{t}) \in \bar{\Omega} \times [0,T]$. By standard arguments, we prove that, up to subsequences, $\bar{x} \to x_0$, $\bar{t} \to 0$, $u(\bar{x},\bar{t}) \to u(x_0,0)$ as $\varepsilon \to 0, \alpha \to 0$. Then for ε, α small enough we can suppose that $\bar{x}$ is enough close to the boundary (i.e. is in the neighbourhood, say V, of the boundary where the distance is smooth (see Remark 4.2.1). Moreover, with some abuse of notations, we denote by $||D^2 d||_\infty$ the supremum of $||D^2 d||_\infty$ over this neighbourhood of the boundary. By Taylor's formula for the distance function, we have for ε, α small enough

$$n(\bar{x})(\bar{x} - x_0) + \frac{1}{2}(\bar{x} - x_0)^T D^2 d(\bar{x})(\bar{x} - x_0) + o(|\bar{x} - x_0|^2) = d(x_0) \geq 0$$

and then

$$n(\bar{x})(\bar{x} - x_0) \geq -||D^2 d||_\infty |\bar{x} - x_0|^2/2 + o(|\bar{x} - x_0|^2). \tag{5.8}$$

Take ε small so that

$$1 + \frac{o(|\bar{x} - x_0|^2)}{|\bar{x} - x_0|^2} \geq 0. \tag{5.9}$$

By (5.8) by the definition of K and (5.9), we have

$$\begin{aligned}\frac{\partial \phi}{\partial n}(\bar{x}) &\geq e^{-Kd(\bar{x})}\frac{|\bar{x}-x_0|^2}{\varepsilon^2}\left[K-||D^2 d||_\infty+\frac{o(|\bar{x}-x_0|^2)}{|\bar{x}-x_0|^2}\right]+\nu-\frac{1}{R}||\psi'||_\infty \\ &\geq e^{-Kd(\bar{x})}\frac{|\bar{x}-x_0|^2}{\varepsilon^2}\left[1+\frac{o(|\bar{x}-x_0|^2)}{|\bar{x}-x_0|^2}\right]+\nu-\frac{1}{R}||\psi'||_\infty>0,\end{aligned}$$

where the last inequality holds by taking $\nu-\frac{1}{R}||\psi'||_\infty\geq 0$. Note that, for $\varepsilon>0$ fixed and ν, R small enough, we can choose $\alpha>0$ small enough so that the viscosity inequality

$$\frac{1}{\alpha}+\mathscr{I}[\phi(\cdot,t)](\bar{x})+H(\bar{x},\bar{t},u(\overline{x,}t),D\phi(\bar{x},\bar{t}))\leq 0$$

cannot hold, since by (E') and (M0) we have for some constant C

$$\mathscr{I}[\phi(\cdot,t)](\bar{x})+H(\bar{x},\bar{t},u(\bar{x},\bar{t}),D\phi(\bar{x},\bar{t}))\geq -C(1+\frac{1}{\varepsilon})-H_R,$$

where H_R is defined in (E') for $R=\max\{|D\phi|,||u||_\infty\}$ (by $|D\phi|$ we denote the supremum of $|D\phi|$ over the neighbourhood V of the boundary where $\bar{x}$ belongs for ε small enough). Note also that $|D\phi|$ depends only on ε. Then, for all ε and α small enough, we have $t=0$ and $u(x,0)\leq u_0(x)$ and by letting $\varepsilon\to 0$, we obtain the desired inequality $u(x_0,0)\leq u_0(x_0)$. The proof of the other inequality is analogous. □

The proof of Theorem 5.2.1 is a consequence of the following Lemma. We remark that the proof is analogous to the proof of Lemma 4.4.1 and Lemma 4.4.8 for the stationary case, with some adaptations to the evolutive setting.

Lemma 5.2.3. *Let $\mathscr{I}$ as in (5.2) and assume μ satisfies (M0), (M1), j satisfies (J0), (J1), (J2). Let H be an Hamiltonian either of Bellman type or a coercive Hamiltonian satisfying (H'). Let $u, v\in L^\infty(\bar{\Omega}\times[0,T])$ for any $T>0$ be respectively usc sub and lsc supersolutions to*

$$\begin{cases}\partial_t w-\mathscr{I}[w(\cdot,t)](x)+H(x,t,w,Dw)=0 & \text{in } \Omega\times(0,+\infty)\\ \frac{\partial w}{\partial n}=0 & \text{on } \partial\Omega\times(0,+\infty),\end{cases}$$

Take $\mu=1$ if H is of Bellman type and if H is sublinearly coercive, $\mu\in(0,1)$ if H is superlinearly coercive. Then, the function

$$\omega(x,t):=\mu u(x,t)-v(x,t)$$

satisfies, in the viscosity sense, the equation

$$\begin{cases}\partial_t\omega+\gamma_R\omega-\mathscr{I}[\omega(\cdot,t)](x)-C_{m,\mu,R}|D\omega|^m\leq A_R(1-\mu) & \text{in } \Omega\times(0,+\infty)\\ \frac{\partial\omega}{\partial n}=0 & \text{on } \partial\Omega\times(0,+\infty),\end{cases}$$

where, for $R = ||\mu u||_\infty + ||v||_\infty$, γ_R is the constant of (H'), $A_R > 0$ is defined in (Hb') and $C_{m,\mu,R} > 0$ is such that $C_{m,\mu,R} = \bar{C}^{1-m}C_R^m 2^{m-1}m^{-m}(1-\mu)^{1-m}$ if $\mu \in (0,1)$ and $C_{m,\mu,R} = \tilde{C}_R$ if $\mu = 1$ and H is sublinearly coercive, where $C_R, \tilde{C}_R$ are defined respectively in $(Hc'), (Ha')$.

Proof. The proof follows closely the arguments presented in Lemma 4.4.1 and Lemma 4.4.8 for the stationary case with some standard adaptations to the evolution setting. The main difference is that now the solution and then the test function depend also on t. We give some details in the case of Hamiltonians in Bellman form and we note that the proof can be extended to the case of coercive Hamiltonians analogously as done in Lemma 4.4.8. The main point we want to highlight is that, in the case of Hamiltonian of Bellman type, we need the uniform continuity also with respect to time as stated in assumption (L') in order to estimate the Hamiltonian terms in the viscosity inequalities satisfied by u and v.

We take $(x_0,t_0) \in \bar{\Omega} \times (0,+\infty)$ and ϕ a smooth function such that $\omega - \phi$ has a strict maximum point at (x_0,t_0). We suppose either $(x_0,t_0) \in \Gamma_{\text{out}}$ or $(x_0,t_0) \in \Gamma$, since the case $(x_0,t_0) \in \Gamma_{\text{in}}$ can be treated analogously as done in Lemma 4.4.1. Let $\varepsilon > 0$. Consider the function

$$\Phi(x,y,t,s) = u(x,t) - v(y,s) - \tilde{\phi}(x,y,t,s) \tag{5.10}$$

where

$$\tilde{\phi}(x,y,t,s) = \phi((x+y)/2,(t+s)/2) + \varepsilon^{-1}\chi_\varepsilon(|x-y|,|t-s|) + K\varepsilon^{-1}|d(x)-d(y)|, \tag{5.11}$$

where d is the signed distance from the boundary (see Remark 4.2.1) and $\chi_\varepsilon : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is defined as follows

$$\chi_\varepsilon(r,p) = \sqrt{r^2+p^2+\varepsilon^4} \quad r,p \in \mathbb{R}. \tag{5.12}$$

By its upper semicontinuity, ϕ attains its maximum over

$$\mathscr{K} := \bar{B}_{2C_j}(x_0) \cap \bar{\Omega} \times \bar{B}_{2C_j}(x_0) \cap \bar{\Omega} \times [0,t_0+1] \times [0,t_0+1]$$

at a point $(\tilde{x},\tilde{y},\tilde{t},\tilde{s})$ and by classical arguments in viscosity theory, we have

$$\bar{x},\bar{y} \to x_0;\ \bar{s},\bar{t} \to t_0;\ \varepsilon^{-1}\chi_\varepsilon(|x-y|,|t-s|) \to 0;\ K\varepsilon^{-1}|d(x)-d(y)| \to 0$$

and

$$u(\bar{x},\bar{s}) \to u(x_0,t_0),\ v(\bar{y},\bar{t}) \to v(x_0,t_0),$$

concluding that for all ε suitably small, $\bar{s},\bar{t} \in (0,t_0+1)$ and $\bar{x},\bar{y} \in \bar{B}_{C_j}(x_0)$.

The proof is carried out step by step analogously as in Lemma 4.4.1. We just remark that in the proof of Lemma 4.4.2 (Step 1 of Lemma 4.4.1), the viscosity inequality satisfied by u (the analogous of (4.58)) is the following

$$\partial_t\tilde{\phi}(\tilde{x},\tilde{y},\tilde{t},\tilde{s}) - \mathscr{I}_\xi[\tilde{\phi}(\cdot,\tilde{y},\tilde{t},\tilde{s})](\tilde{x}) - \mathscr{I}^\xi[u(\cdot,\tilde{t})](\tilde{x}) + H(\tilde{x},\tilde{t},u(\tilde{x},\tilde{t}),D_x\tilde{\phi}(\tilde{x},\tilde{y},\tilde{t},\tilde{s})) \leq 0,$$

where we use the notation $D_x\tilde{\phi}(\tilde{x},\tilde{y},\tilde{t},\tilde{s}) = D_x[\tilde{\phi}(\cdot,\tilde{y},\tilde{t},\tilde{s})](\tilde{x})$ and analogously $\partial_t\tilde{\phi}(\tilde{x},\tilde{y},\tilde{t},\tilde{s}) = \partial_t[\tilde{\phi}(\tilde{x},\tilde{y},\cdot,\tilde{s})](\tilde{t})$. Note that the additional term $\partial_t\tilde{\phi}$ plays the same role of the Hamiltonian term, since, by the smoothness of ϕ and

$$0 \le \partial_p\chi_\varepsilon(|\tilde{x}-\tilde{y}|,|\tilde{t}-\tilde{s}|) \le 1, \tag{5.13}$$

we have

$$\partial_t\tilde{\phi}(\tilde{x},\tilde{y},\tilde{t},\tilde{s}) = \partial_t\phi((\tilde{x}+\tilde{y})/2,(\tilde{t}+\tilde{s})/2)/2 + \varepsilon^{-1}\partial_p\chi_\varepsilon(|\tilde{x}-\tilde{y}|,|\tilde{t}-\tilde{s}|)\hat{r} \ge C - \varepsilon^{-1},$$

where $C > 0$ is a constant independent of ε and

$$\hat{r} = \frac{(\tilde{t}-\tilde{s})}{|\tilde{t}-\tilde{s}|}.$$

Then, we conclude the proof of Lemma 4.4.2 analogously, by applying Lemma 5.3.9 (see Appendix C) and Lemma 4.4.3 to estimate respectively the Hamiltonian term and the nonlocal terms. Analogously as in Lemma 4.4.1, Step 2, we regularize the test function and we consider

$$\tilde{\phi}(x,y,t,s) = \phi((x+y)/2,(t+s)/2) + \varepsilon^{-1}\chi_\varepsilon(|x-y|,|t-s|) + K\varepsilon^{-1}\chi_\delta(|d(x)-d(y)|),$$

where d is the signed distance from the boundary (see Temark 4.2.1) χ_ε is defined as in (5.12) and χ_δ is defined as follows

$$\chi_\delta(r) = \sqrt{r^2+\delta^4}, \quad r \in \mathbb{R}.$$

We denote by $(\bar{x},\bar{y},\bar{t},\bar{s})$ a maximum point of $u(x)-v(y)-\tilde{\phi}(x,y,t,s)$ on $\mathscr{K}$. We write the viscosity inequalities for u and v and by (H'), we get

$$\begin{aligned}\gamma_R(u(\bar{x},\bar{t})-v(\bar{y},\bar{s})) &\le H(\bar{y},\bar{s},v(\bar{y},\bar{s}),-D_y[\tilde{\phi}(\bar{x},\bar{y},\bar{t},\bar{s}]) - H(\bar{x},\bar{t},v(\bar{y},\bar{s}),D_x[\tilde{\phi}(\bar{x},\bar{y},\bar{t},\bar{s})])\\ &\quad + \mathscr{I}^{\xi'}[u(\cdot,\bar{t})](\bar{x}) - \mathscr{I}^{\xi'}[v(\cdot,\bar{s})](\bar{y}) + \mathscr{I}_{\xi'}[\tilde{\phi}(\cdot,\bar{s},\bar{y},\bar{s})](\bar{x}) - \mathscr{I}_{\xi'}[-\tilde{\phi}(\bar{x},\cdot,\bar{t},\bar{s})](\bar{y}).\end{aligned}$$

where γ_R is defined in (H') for $R = ||u||_\infty + ||v||_\infty$. Note that the dependence on t does not play any role in the estimation of the nonlocal terms. Note also that, in the case of Hamiltonian of Bellman type, we use assumption (L') to conclude

$$\begin{aligned}&H(\bar{y},\bar{s},v(\bar{y},\bar{s}),D_x[\tilde{\phi}(\bar{x},\bar{y},\bar{t},\bar{s})] - H(\bar{x},\bar{t},v(\bar{y},\bar{s}),D_x[\tilde{\phi}(\bar{x},\bar{y},\bar{t},\bar{s})])\\ &\le C\left(\varepsilon^{-1}(|\bar{x}-\bar{y}|+|\bar{t}+\bar{s}|)+|\bar{x}-\bar{y}|+|\bar{t}+\bar{s}|\right)+\omega_l(|\bar{x}-\bar{y}|+|\bar{t}-\bar{s}|)+\omega_\lambda(|\bar{x}-\bar{y}|+|\bar{t}-\bar{s}|).\end{aligned} \tag{5.14}$$

The remaining terms are treated as in Lemma 4.4.1, in particular we have

$$H(\bar{y},\bar{s},v(\bar{y},\bar{s}),-D_y[\tilde{\phi}(\bar{x},\bar{y},\bar{t},\bar{s})])-H(\bar{y},\bar{s},v(\bar{y},\bar{s}),D_x[\tilde{\phi}(\bar{s},\bar{y},\bar{t},\bar{s})]$$
$$\leq B\left(|D\phi((\bar{x}+\bar{y})/2,(\bar{t}+\bar{s})/2)|+K\varepsilon^{-1}|\bar{x}-\bar{y}|\right) \quad (5.15)$$

where $B=\sup_{x\in\bar{\Omega},\alpha\in\mathscr{A}} b(x,\alpha)$. Then coupling (5.14) and (5.15) we get

$$H(\bar{y},\bar{s},v(\bar{y},\bar{s}),-D_y[\tilde{\phi}(\bar{x},\bar{y},\bar{t},\bar{s})])-H(\bar{x},\bar{t},v(\bar{y},\bar{s}),D_x[\tilde{\phi}(\bar{x},\bar{y},\bar{t},\bar{s})])$$
$$\leq B|D\phi((\bar{x}+\bar{y})/2,(\bar{t}+\bar{s})/2)|+o_{\delta,\varepsilon}(1),$$

where $o_{\delta,\varepsilon}(1)$ means $\lim_{\delta\to 0} o_{\delta,\varepsilon}(1)=o_\varepsilon(1)$. Then we conclude the proof as in Lemma 4.4.1. □

Now we sketch the proof of Theorem 5.2.1. We give the details only in the case of Hamiltonians of Bellman type, since for Hamiltonians in coercive form the proof is analogous.

Proof of Theorem 5.2.1. Let $T>0$. We argue over the finit horizon problem (5.3), from which the general result follows by the fact that T is arbitrary. We observe that the proof is analogous to that of Theorem 4.2.6, except for the fact that, in principle, we have we deal also with the part of the parabolic boundary $\{0\}\times\bar{\Omega}$. We sketch the proof in order to show that this extra difficulty can be easily treated essentially through Lemma 5.2.2. We assume by contradiction that

$$M:=\sup_{\bar{\Omega}\times[0,T]}\{u-v\}>0,$$

we denote $\omega(x,t)=u(x,t)-v(x,t)\quad (x,t)\in\bar{\Omega}\times[0,T]$ and consider the function

$$\Phi(x,t)=\omega(x,t)-\psi(R^{-1}|x|)+\nu d(x)-\eta t, \quad (x,t)\in\bar{\Omega}\times[0,T]$$

where ψ is a smooth function as in (4.125) and d as in Remark 4.2.1, in particular coincides with the distance from the boundary of Ω in a neighboorhood of the boundary and it is extended bounded in all the domain. By similar arguments as in Theorem 4.2.6, we show that the function Φ achieves its positive maximum at a point (x,t) for R large and ν,η small enough.

In particular, by Lemma 5.2.2 and taking R bigger and ν,η smaller if it is necessary, we have that $t>0$ for all such parameters. Indeed if $t=0$ we would get

$$\max\Phi=\Phi(x,0)=u(x,0)-v(x,0)-\psi(R^{-1}|x|)+\nu d(x)\leq\nu d(x)$$

which tends to zero if $\nu\to 0$ getting in contradiction with the positivity of the maximum of Φ for all R large enough and ν small enough. At this point, we fix $\eta>0$ satisfying the above facts and we continue the proof analogously as in Theorem 4.2.6. □

For both the coercive and Bellman case, the application of Perron's method on a sequence of finite-time horizon problems with the form (5.3) with $T \to +\infty$ and the strong comparison principle allows us to get the existence of a solution which is defined for all time.

In order to apply Perron's method, we ask the initial datum to be bounded, i.e. $u_0 \in BC(\Omega)$. We refer to the following proof for further details.

For reasons which will be clear in the next theorem, we assume the following assumption:

(H") There exists $\gamma_0 > 0$ such that $\gamma_R \geq \gamma_0$ for all $R > 0$ where γ_R is defined in (H').

Theorem 5.2.4. *Let Ω be an open subset of $\mathbb{R}^n$ satisfying $(O), u_0 \in BC(\mathbb{R}^n)$. Assume (M0),(M1),(J0),(J1),(J2) and let H be an Hamiltonian of Bellman type or a coercive Hamiltonian satisfying (H') and (E'). Then, there exists a unique viscosity solution to problem* (5.1) *in $C(\bar{\Omega} \times [0,+\infty)) \cap L^\infty(\bar{\Omega} \times [0,T])$. In addition, if (H") holds, then the unique solutions $u \in C(\bar{\Omega} \times [0,+\infty)) \cap L^\infty(\bar{\Omega} \times [0,T])$ for all $T > 0$ is uniformly bounded in $\bar{\Omega} \times [0,+\infty)$.*

Proof. The uniqueness follows from Theorem 5.2.1. In order to prove the esistence, we apply Perron's method on a sequence of finite-time horizon problems with the form (5.3) with $T \to +\infty$. The role of the global sub and supersolutions present in the Perron's method is played by functions with the form $(x,t) \to c_1 t + c_2$, for suitable constants c_1, c_2 depending on the data, whose construction is carried out through assumptions (H') and (E'). We give the details in the case of the the supersolution (for the subsolution is analogous). Let $g(t) := c_1 t + c_2$. We want to find c_1, c_2 such that g is a supersolution of the problem (5.1). We take

$$c_2 \geq ||u_0||_\infty.$$

Note that by (H') and (E') we have

$$\partial_t g + H(x,t,g(t),Dg(t)) = c_1 + H(x,t,c_1 t + c_2, 0) \geq c_1 + \gamma_R(c_1 t + c_2) - H_0,$$

where $\gamma_R \geq 0$ is defined in (H') for $R = \max\{c_1 T + c_2, T\}$ and H_0 is defined in E' for $R = 0$. Then we conclude the proof by taking $c_1 \geq H_0$. Note that H_0 is independent of T since we assumed the boundedness of l and f respectively for Hamiltonians in Bellman form and for coercive Hamiltonians.

Moreover, under the assumption (H"), these global sub and supersolutions can be taken as constant functions depending on the data, but not on T, concluding the uniform boundedness. Indeed, we take $g = c$ with $c \geq ||u_0||_\infty$ and we note that by (H") we have

$$\partial_t g + H(x,t,g(t),Dg(t)) = H(x,t,c,0) \geq \gamma_0 c - H_0,$$

and we conclude by taking $c \geq H_0 \gamma_0^{-1}$. □

5.3 Large time behavior

5.3.1 Large time behavior I: convergence in the classical sense

In the next subsections we address the question of the asymptotic behavior of the solution as $t \to \infty$ which naturally arises once the existence and uniqueness for problem (5.1) is obtained. We remark that the same kind of results have been given in the case of the Dirichlet problem for nonlocal equation (fractional laplacian) both with coercive Hamiltonians and of Bellman type by E. Topp in [34]. Note that asumption (H") allows us to get the strong comparison principle and therefore existence and uniqueness for the associated stationary problem.

Theorem 5.3.1. *Let Ω be an open subset of $\mathbb{R}^n$ satisfying $(O), u_0 \in BC(\Omega)$ and H an Hamiltonian either of Bellman type or a coercive Hamiltonian satisfying (H"). Assume (E'), (M0), (M1), (J0), (J1), (J2). Assume also that there exists a continuous function $\bar{H} : \bar{\Omega} \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ satisfying*

$$H(\cdot, t, \cdot, \cdot) \to \bar{H} \qquad \textit{locally uniformly in } \bar{\Omega} \times \mathbb{R} \times \mathbb{R}^n,$$

as $t \to \infty$. Then, there exists a unique bounded viscosity solution u for the following problem

$$\begin{cases} -\mathscr{I}(u) + \bar{H}(x, u, Du) = 0 & \textit{in } \Omega \\ \frac{\partial u}{\partial n} = 0 & \textit{on } \partial\Omega. \end{cases} \tag{5.16}$$

Moreover, the unique viscosity solution u of (5.1) *converges uniformly on compact sets in $\bar{\Omega}$ to u_∞, the unique viscosity solution of the problem* (5.16)*.*

Proof. For the existence and uniqueness for the problem (5.16) we refer to Theorem 4.2.6 and Corollary 4.2.7 of Chapter 4. The proof of the convergence of the solution to (5.1) to the unique solution of (5.16) is rather classical and uses the relaxed semi-limits technique. In particular, for each $(x,t) \in \bar{\Omega} \times [0, +\infty)$, define the functions

$$\bar{u}(x,t) = \limsup_{\varepsilon \to 0, z \to x, z \in \Omega} u(z, \frac{t}{\varepsilon}), \quad \underline{u}(x,t) = \liminf_{\varepsilon \to 0, z \to x, z \in \Omega} u(z, \frac{t}{\varepsilon}),$$

which are well defined by the uniform boundedness of u. The application of the half-relaxed semilimits method proves that for all $t > 0$, the functions $x \to \bar{u}(x,t)$ and $x \to \underline{u}(x,t)$ are respectively viscosity sub and supersolution for problem (5.16). Then, by comparison principle we have $\bar{u} = \underline{u}$ in $\bar{\Omega} \times [0, +\infty)$ and consequently $\bar{u}(x,t) = \underline{u}(x,t) = u_\infty(x)$ for all $(x,t) \in \bar{\Omega} \times [0, +\infty)$ by the uniqueness of (5.16). This conclude the proof. $\square$

5.3.2 Large time behavior II: convergence to the ergodic problem

In this subsection we prove large time behavior for the problem

$$\begin{cases} \partial_t u(x) - \mathscr{I}[u(\cdot,t)](x) + H(x,Du) = 0 & \text{in } \Omega \times (0,+\infty) \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega \times (0,\infty), \\ u(x,0) = u_0(x) & \text{in } \bar{\Omega}. \end{cases} \tag{5.17}$$

where Ω is a bounded open subset of $\mathbb{R}^n$ satisfying (O), $u_0 \in C(\Omega)$ and H is an Hamiltonian in superfractional coercive form, that is when $m > 1$ in (H1').

Existence and uniqueness for the problem (5.17) follow from Theorem 5.2.4. Note that H does not depend on u, so it does not satisfy (H').

The main result of this section is Theorem 5.3.7, namely the convergence as $t \to +\infty$ of the solution of (5.17) to a solution of the ergodic problem, which we solve in Proposition 5.3.6. We follow the methods of [29]. In particular, we rely on the Hölder regularity up to the boundary and a control of the oscillation of subsolutions, which has been proved by Barles, Ley Koike, Topp in [29] (see also Barles and Topp [33]), in the case of censored operators and coercive Hamiltonian with $m > 1$.

We remark that, differently from [32], where Lipschitz regularity of the solutions is used to linearize the equations in order to apply the Strong Maximum Principle, our proof relies mainly on the use of a Strong Maximum Principle à la Coville [64], [65]) (see also Ciomaga [58]). This means that it relies mainly on a topological property of the the support of the measure defining the nonlocal operator. Note that in this final part we assume Ω bounded for techincal reasons related to the proof of the Strong maximum principle, we refer to the proof of Proposition 5.3.2.

A strong maximum principle

We need some notation for the statement of the Strong Maximum Principle. Let μ, j be as in the definition of the nonlocal operator $\mathscr{I}$, that is satisfying (M0), (M1), (J0), (J1), (J2) and denote by $\text{supp}\mu$ the support of the measure μ.

For $x \in \mathbb{R}^n$ we define inductively

$$X_0(x) = \{x\}; \quad X_{r+1}(x) = \cup_{\xi \in X_r(x)} \{\xi + j(\xi, \text{supp}\{\mu_x\})\} \cap \bar{\Omega}, \quad \text{for } r \in \mathbb{N},$$

and

$$\mathscr{X}(x) = \overline{\cup_{r\in\mathbb{N}} X_r}.$$

The Strong Maximum Principle presented in this paragraph relies in the nonlocality of the operator under the "iterative covering property"

$$\mathscr{X}(x) = \Omega, \quad \text{for all } x \in \Omega. \tag{5.18}$$

The most basic example is the case where $j(x,z) = z$ and there exists $r > 0$ such that $B_r \subset \text{supp}\{\mu\}$. For further details and examples we refer to [29].

The following proposition states the Strong Maximum Principle.

Proposition 5.3.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain satisfying (O). Let H be an Hamiltonian in coercive form with $m > 1$ in (H1'). Assume (M0), (M1), (J0), (J1), (J2), (E') and*(5.18)*. Let u, v be respectively a sub and a supersolution of* (5.17)*, such that there exists $(x_0, t_0) \in \bar{\Omega} \times (0, +\infty)$ satisfying*

$$(u - v)(x_0, t_0) = \sup_{\bar{\Omega} \times (0,+\infty)} \{u - v\}.$$

Then, the function $u - v$ is constant in $\bar{\Omega} \times [0, t_0]$. Moreover we have

$$(u - v)(x,t) = \sup_{x \in \bar{\Omega}} \{u(x,0) - v(x,0)\}, \textit{ for all } (x,t) \in \bar{\Omega} \times [0, t_0].$$

The proof of Proposition 5.3.2 uses the following lemma, which is is a consequence of the comparison principle, see [32], Theorem 4.1.

Lemma 5.3.3. *Let assumptions of Proposition 5.3.2 hold. Let u, v be locally bounded sub and supersolution to equation* (5.17) *and for $t \in [0, +\infty)$ define*

$$k(t) = \max_{\bar{\Omega}} \{u(x,t) - v(x,t)\}.$$

Then, for all $0 \le s \le t$, we have $k(t) \le k(s)$.

We prove the strong maximum principle. We follow the argument given in [29], Proposition 4.1, with some changes due to the Neumann condition on the boundary.

Proof of Proposition 5.3.2. We divide the proof into several steps.

Step. *1-Preliminaries* We want to prove that for each $(x,t) \in \bar{\Omega} \times [0, t_0]$

$$(u - v)(x,t) = k(0).$$

Since $k(t_0)$ is a global maximum value of k in $[0, +\infty)$, by Lemma 5.3.3 we have $k(t) = k(0)$ for all $t \in [0, t_0]$. Then, we have just to prove that

$$u(x, \tau) - v(x, \tau) = k(\tau), \ \forall x \in \bar{\Omega},$$

for each $\tau \in (0, t_0)$. Then, by upper-semicontinuity, we derive the result up to $\tau = 0$ and $\tau = t_0$.

Fix $\tau \in (0,t_0)$ and define the set

$$\mathscr{B}_\tau = \{x \in \bar{\Omega} : (u-v)(x,\tau) = k(\tau)\}. \tag{5.19}$$

We observe that by the upper-semicontinuity of $u-v$, $\mathscr{B}_\tau$ is nonempty. Then, the claim of the proposition follows once proved that $\mathscr{B}_\tau = \bar{\Omega}$.

Step. *2-Localization on time* τ For $\eta > 0$, define the function

$$(x,t) \to \Phi(x,t) := u(x,t) - v(x,t) - \eta(t-\tau)^2$$

and note that for each $(x,t) \in \bar{\Omega} \times (0,+\infty)$ and for $\tilde{x} \in \mathscr{B}_\tau$ where $\mathscr{B}_\tau$ is defined in (5.19), we have

$$\Phi(x,t) \le k(t) - \eta(t-\tau)^2 \le k(\tau) = (u-v)(\tilde{x},\tau) = \bar{W}(\tilde{x},\tau).$$

Then the supremum of Φ in $\bar{\Omega} \times (0,+\infty)$ is achieved and

$$\sup_{(x,t)\in\Omega\times(0,+\infty)} \Phi(x,t) = k(\tau).$$

Step. *3-Localization around a point in* $\mathscr{B}_\tau$ From now on we fix $x_\tau \in \mathscr{B}_\tau$ where $\mathscr{B}_\tau$ is defined in (5.19). We define for $\varepsilon, \alpha > 0$

$$\psi_{\varepsilon,\alpha}(x) = e^{-Kd(x)} \frac{|x-x_\tau|^2}{\varepsilon^2} - \alpha d(x),$$

where d is the signed distance from the boundary (see Remark 4.2.1) and $K > 0$ is a constant satisfying

$$K > ||D^2 d||_\infty + 1, \tag{5.20}$$

where, with some abuse of notations, we denote by $||D^2 d||_\infty$ the supremum of $||D^2 d||_\infty$ over the neighbourhood of the boundary where d is smooth. We observe that

$$\psi_{\varepsilon,\alpha}(x_\tau) = -\alpha d(x_\tau) \tag{5.21}$$

and the first derivatives of $\psi_{\varepsilon,\alpha}$ are bounded, depending on ε and α.

For $0 < \mu < 1$ we denote

$$\omega_\mu = \mu u - v \tag{5.22}$$

and we consider

$$(x,t) \to \Phi_\mu(x,t) := \omega_\mu(x,t) - \eta|t-\tau|^2 - (1-\mu)\psi_{\varepsilon,\alpha}(x).$$

Then, by the upper-semicontinuity of Φ_μ, there exists $(x_\mu, t_\mu) \in \bar{\Omega} \times [0, t_0 + 1]$ such that

$$\Phi_\mu(t_\mu, x_\mu) = \max_{\bar{\Omega} \times [t_0, t_0+1]} \Phi_\mu.$$

Since $\Phi_\mu \to \Phi$ locally uniformly on $\bar{\Omega} \times [0, +\infty)$ as $\mu \to 1$, we have that, up to subsequences,

$$(x_\mu, t_\mu) \to (\bar{x}, \tau) \quad \text{as } \mu \to 1.$$

Not also that for any α small enough small

$$\bar{x} = \bar{x}_\varepsilon \to x_\tau \quad \text{as } \varepsilon \to 0.$$

Indeed, by using the maximum point inequality for Φ_μ, we have

$$\begin{aligned} \Phi_\mu(x_\mu, t_\mu) &= (u - v)(x_\mu, t_\mu) + (\mu - 1)(u + \psi_{\varepsilon,\alpha})(x_\mu, t_\mu) - \eta |t_\mu - \tau|^2 \\ &\geq k(\tau) + (\mu - 1)u(x_\tau, \tau) - \alpha(\mu - 1)d(x_\tau) \end{aligned} \tag{5.23}$$

where we used the definition of $k(\tau)$ and (5.21). Since $t_\mu \in [t_0, t_0 + 1]$ for all μ close to 1, we have

$$(u - v)(x_\mu, t_\mu) \leq k(t_\mu) \leq k(\tau)$$

and coupling the previous inequality with (5.23) we get

$$\psi_{\varepsilon,\alpha}(x_\mu) + \alpha d(x_\tau) \leq u(x_\tau, \tau) - u(x_\mu, t_\mu)$$

and then, by the boundedness of u and of d and for α small, we deduce that

$$|x_\mu - x_\tau| \leq C\varepsilon \tag{5.24}$$

for some $C > 0$ independent on μ. Then we deduce that for any α enough small

$$\bar{x} \to x_\tau \quad \text{as } \varepsilon \to 0.$$

Step. *4-Writing the viscosity inequality for* ω_μ Denote

$$\phi(x, t) := (1 - \mu)\psi_{\varepsilon,\alpha}(x) + \eta(t - \tau)^2$$

We test ω_μ defined in (5.22) with the function ϕ in (x_μ, t_μ). We suppose $x_\mu \in \partial\Omega$, since the other case being analogous and even simpler. By the Taylor's formula for the distance function, we have

$$n(x_\mu)(x_\mu - x_\tau) + \frac{1}{2}(x_\mu - x_\tau)^T D^2 d(x_\mu)(x_\mu - x_\tau) + o(|x_\mu - x_\tau|^2) = d(x_\tau) \geq 0$$

and then

$$n(x_\mu)(x_\mu - x_\tau) \geq -||D^2 d||_\infty |x_\mu - x_\tau|^2/2 + o(|x_\mu - x_\tau|^2). \tag{5.25}$$

Take μ, ε small enough so that

$$1 + \frac{o(|x_\mu - x_\tau|^2)}{|x_\mu - x_\tau|^2} \geq 0 \tag{5.26}$$

By (5.25), by the definition (5.20) of K and (5.26), we have

$$\begin{aligned}
\frac{\partial \phi_{\varepsilon,\alpha}}{\partial n}(x_\mu, t) &= \frac{\partial \psi_{\varepsilon,\alpha}}{\partial n}(x_\mu) \\
&\geq e^{-Kd(x_\mu)} \frac{|x_\mu - x_\tau|^2}{\varepsilon^2} \left[K - ||D^2 d||_\infty + \frac{o(|x_\mu - x_\tau|^2)}{|x_\mu - x_\tau|^2} \right] + \alpha \\
&\geq e^{-Kd(x_\mu)} \frac{|x_\mu - x_\tau|^2}{\varepsilon^2} \left[1 + \frac{o(|x_\mu - x_\tau|^2)}{|x_\mu - x_\tau|^2} \right] + \alpha > 0.
\end{aligned}$$

Then, by Lemma 5.2.3, we get for $\xi > 0$

$$\begin{aligned}
2\eta(t_\mu - \tau) - \mathscr{I}^\xi[\omega_\mu(\cdot, t_\mu)](x_\mu) - \mathscr{I}_\xi[(1-\mu)\psi_{\varepsilon,\alpha}(\cdot)](x_\mu) \\
- (1-\mu) C_{\mu,m} |D\psi_{\varepsilon,\alpha}(x_\mu)|^m) \leq A_R(1-\mu),
\end{aligned}$$

where $C_{\mu,m}$ is defined in Lemma 5.2.3 and A_R (also ariseing in Lemma 5.2.3) is defined in (Hb'). Since

$$\mathscr{I}_\xi[(1-\mu)\psi_{\varepsilon,\alpha}(\cdot)](x_\mu) \leq (1-\mu) C' ||D\psi_{\varepsilon,\alpha}||_\infty$$

for some $C' > 0$, we get

$$\begin{aligned}
2\eta(t_\mu - \tau) - \mathscr{I}^\xi[\omega_\mu(\cdot, t_\mu)](x_\mu) \\
- (1-\mu)\left(C' ||D\psi_{\varepsilon,\alpha}||_\infty + (1-\mu) C_{\mu,m} |D\psi_{\varepsilon,\alpha}(x_\mu)|^m) + A_R\right) \leq 0.
\end{aligned} \tag{5.27}$$

We recall that $t_\mu \to \tau$ as $\mu \to 1$ and we observe that by the smoothness of $\psi_{\varepsilon,\alpha}$ the term in parenthesis in (5.27) remain bounded as $\mu \to 1$. Moreover, by the Dominated Convergence Theorem, we get

$$\mathscr{I}^\xi[\omega_\mu(\cdot, t_\mu)](x_\mu) \to \mathscr{I}^\xi[(u-v)(\cdot, \tau)](\bar{x}) \text{ as } \mu \to 1$$

where we recall that $\bar{x}$ is the limit of x_μ as $\mu \to 1$. Then

$$\int_{\substack{\bar{x} + j(\bar{x},z) \in \bar{\Omega}, \\ |z| \geq \xi}} (u-v)(\bar{x} + j(\bar{x},z), \tau) - (u-v)(\bar{x}, \tau) d\mu_{\bar{x}}(z) = 0$$

and letting $\varepsilon \to 0$ and recalling that $\bar{x} \to x_\tau$ as $\varepsilon \to 0$ and $(u-v)(x_\tau, \tau) = k(\tau)$ we finally conclude

$$\int_{\substack{x_\tau + j(x_\tau, z, \tau) \in \bar{\Omega} \\ |z| \geq \xi}} (u-v)(x_\tau + j(x_\tau, z)) - k(\tau) d\mu_{x_\tau}(z) = 0.$$

Since $\xi > 0$ is arbitrary, we get that

$$(u-v)(x,\tau) - k(\tau) = 0 \quad \text{for all } x \in X_1(x_\tau).$$

Therefore we can proceed in the same way as above, and conclude by induction that

$$(u-v)(x,\tau) - k(\tau) = 0 \quad \text{for all } x \in \cup_{r \in \mathbb{N}} X_r(x_\tau).$$

Then we conclude the proof by the upper-semicontinuity of $u - v$ and applying the iterative convering property (5.18).

□

The ergodic problem

Roughly speaking, solving the *ergodic problem* means pass to the limit as $\delta \to 0$ in the stationary problem

$$\begin{cases} \delta u(x) - \mathscr{I}[u(\cdot)](x) + H(x, Du) = 0 & \text{in } \Omega \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.28}$$

whose existence and uniqueness for $\delta > 0$ holds by Theorem 4.2.6. In particular, the following proposition holds.

Proposition 5.3.4. *Let $\delta > 0$, $\Omega \subset \mathbb{R}^n$ be a bounded domain satisfying (O). Let H be an Hamiltonian in coercive form with $m > 1$ in (H1'). Assume (M0),(M1),(J0),(J1),(J2), (E'). Then*

(i) If u, v are bounded viscosity sub and supersolution to equation (5.28)*, then $u \leq v$ in $\bar{\Omega}$.*
(ii) There exists a unique viscosity solution $u \in BC(\Omega)$ to (5.28) *which satisfies*

$$||u||_\infty \leq \delta^{-1} ||H(\cdot, 0)||_\infty.$$

In order to solve the ergodic problem we need the compactness of the familty of solutions $\{u_\delta\}$, which relies mainly on the regularity result for subsolutions of equation (5.28) proved in [29], Theorem 5.5 and which we recall in the following proposition.

Proposition 5.3.5. *Let the assumptions of Proposition 5.3.4 hold. Then any bounded viscosity subsolution $u : \mathbb{R}^n \to \mathbb{R}$ to* (5.28) *is Hölder continuous in $\bar{\Omega}$ with Hölder exponent $\gamma_0 = \frac{m-\sigma}{m}$ and Hölder seminorm depending on Ω, the data and $osc_\Omega(u)$ and not on δ.*

Moreover, there exists $K > 0$ such that for any bounded viscosity subolsoution of (5.28) *we have*

$$osc_\Omega(u) \leq K. \tag{5.29}$$

Proposition 5.3.6. *Under the assumptions of Proposition 5.3.4 and the additional assumption* (5.18)*, there exists a unique constant $\lambda \in \mathbb{R}$ for which the stationary ergodic problem*

$$\begin{cases} \lambda - \mathscr{I}[u(\cdot)](x) - H(x,Du) = 0 & in\ \Omega \\ \frac{\partial u}{\partial n} = 0 & in\ \partial\Omega \end{cases} \tag{5.30}$$

has a solution $w \in C^{\frac{m-\sigma}{m}}(\bar{\Omega})$. Moreover w is the unique solution of (5.30) *up to an additive constant.*

Proof. We consider the stationary problem (5.28) for $\delta > 0$ and its unique solution w_δ, which exists and satisfies the estimate

$$||w_\delta||_\infty \leq \delta^{-1}||H(\cdot,0)||_\infty,$$

by Proposition 5.3.4. Moreover, thanks to Proposition 5.3.5 we have that $w_\delta \in C^{\frac{m-\sigma}{m}}(\bar{\Omega})$, with Hölder seminorm independent of δ or $||w_\delta||_\infty$. Define $v_\delta = w_\delta - w_\delta(0)$ and observe that v_δ satisfies

$$\begin{cases} \delta w_\delta(0) + \delta v_\delta(x) - \mathscr{I}[v_\delta(\cdot)](x) + H(x,Dv_\delta) = 0 & \text{in } \Omega \\ \frac{\partial u_\delta}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases}$$

By Proposition 5.3.5 we have that $\{v_\delta\}_{\delta\in(0,1)}$ is uniformly bounded and equi-Hölder with exponent $\frac{m-\sigma}{m}$. Then, by Ascoli-Arzelà Theorem, there exist $w \in C^{\frac{m-\sigma}{m}}(\bar{\Omega})$ such that $v_\delta \to w$ as $\delta \to 0$ uniformly on $\bar{\Omega}$ and a constant $\lambda \in \mathbb{R}$ such that $\delta w_\delta(0) \to \lambda$ as $\delta \to 0$. We conclude that (w,λ) is a solution of (5.30) by the stability result for viscosity solutions (see for example [5], [67]).

Now we prove the uniqueness. Suppose that $(w_i,\lambda_i)_{i=1,2}$ are two solutions of (5.30), then $u_i(x,t) = w_i(x) + \lambda_i t$ for $i = 1,2$ are two solutions to the Cauchy problem (5.17) with initial data w_i. Then, thanks to the comparison principle we get

$$u_1(x,t) - ||w_1 - w_2||_\infty \leq u_2(x,t), \text{ for all } (x,t) \in \bar{\Omega} \times [0,+\infty),$$

and then

$$(\lambda_1 - \lambda_2)t \leq 2||w_1 - w_2||_\infty.$$

We divide by t and we let $t \to \infty$ and obtain

$$\lambda_1 \leq \lambda_2.$$

Similarly, echanging the role of w_1 and w_2, we get

$$\lambda_1 = \lambda_2 = \lambda$$

and then we conclude the uniqueness of λ. The uniqueness up to and additive constant follows from the application of the strong maximum principle for the problem (5.17) (see Proposition 5.3.2). Indeed, for any $t \in [0, +\infty)$, we have

$$\sup_{\bar{\Omega}}\{u_1(x,t) - u_2(x,t)\} = \sup_{\bar{\Omega}}\{w_1 - w_2\} =: M$$

and then we finally conclude by Proposition 5.3.2 that

$$w_1(x) = w_2(x) + M \text{ for each } x \in \bar{\Omega}.$$

$\square$

Convergence as $t \to +\infty$

Theorem 5.3.7. *Let assumptions of Proposition 5.3.4 and the additional assumption* (5.18) *hold. Let u be the unique solution to problem* (5.17)*. Then, there exists a pair (w, λ) solution to* (5.30) *such that*

$$u(x,t) - \lambda t - w(x) \to 0 \text{ as } t \to +\infty,$$

uniformly on $\bar{\Omega}$.

We follow closely the arguments given in [29] (see also [23], [34] for the local framework and [151] for the nonlocal one). A crucial point is the Hölder regularity of the solutions of (5.17) proved in [29], Theorem 5.5 and Theorem 5.6, which we recall in Lemma 5.3.8. Once established the regularity, the proof of Theorem 5.3.7 is very similar to that of Theorem 5.6 of [29].

Lemma 5.3.8. *Let u be the viscosity subsolution of* (5.17) *with $u_0 \in C^2(\Omega)$. Then u is in $C^{\frac{m-\sigma}{m},1}(\bar{\Omega} \times (0, +\infty))$.*

Now we prove Theorem 5.3.7.

Proof of Theorem 5.3.7. Let first $u_0 \in C^2(\bar{\Omega})$. Then, by applying Lemma 5.3.8, we get that the unique solution u to problem (5.17) is in $C^{\frac{m-\sigma}{m},1}(\bar{\Omega} \times (0, +\infty))$. Moreover, we observe that u and the function $(x,t) \to w(x) + \lambda t$ are solutions of (5.17). Then, by the comparison principle we have

$$||u(\cdot,t) - w - \lambda t||_\infty \le ||u_0 - w||_\infty. \tag{5.31}$$

We define the function $(x,t) \to v(x,t) := u(x,t) - \lambda t$. Thanks to (5.31), we get for each $t \ge 0$,

$$||v(\cdot,t)||_\infty \le ||w||_\infty + ||u_0 - w||_\infty.$$

Since the family $\{v(\cdot,t)\}_t$ is equi-Hölder with exponent $\frac{m-\sigma}{m}$, by Ascoli-Arzelà Theorem, we extract a subsequence $\{v(\cdot,t_k)\}_k$ with $t_k \to +\infty$ as $k \to +\infty$ and

$$v(\cdot,t_k) \to \overline{v} \text{ as } k \to +\infty \quad \text{uniformly in } \bar{\Omega}.$$

Let $v_k(x,t) = v(x,t+t_k)$ and observe that v_k is solution of

$$\begin{cases} \lambda + \partial_t v_k - \mathscr{I}[v_k(\cdot,t)](x) + H(x,Dv_k) = 0 & \text{in } \Omega \times (0,+\infty) \\ \frac{\partial v_k}{\partial n} = 0 & \text{on } \partial\Omega. \\ v_k(x,0) = v(x,t_k) & x \in \Omega \end{cases}$$

then by means of the comparison principle we conclude, for all $t \ge 0$ and $k_1, k_2 \in \mathbb{N}$,

$$||v_{k_1} - v_{k_2}||_{L^\infty(\bar{\Omega}\times(0,+\infty))} \le ||v(\cdot,t_{k_1}) - v(\cdot,t_{k_2})||_\infty.$$

Then since $\{v_k\}_k$ is an uniformly bounded Cauchy sequence in $C(\bar{\Omega} \times (0,+\infty))$ we obtain, up to subsequence, that $v_k \to \widetilde{v}$ in $C(\bar{\Omega} \times (0,+\infty))$ as $k \to +\infty$, where $\widetilde{v}$ is a solution of

$$\begin{cases} \lambda + \partial_t \widetilde{v} - \mathscr{I}[\widetilde{v}(\cdot,t)](x) + H(x,D\widetilde{v}) = 0 & \text{in } \Omega \times (0,+\infty) \\ \frac{\partial \widetilde{v}}{\partial n} = 0 & \text{on } \partial\Omega \\ \widetilde{v}(x,0) = \overline{v} & x \in \Omega. \end{cases}$$

Moreover, we observe that, by applying Lemma 5.3.3 to $K(t)$ defined as follows

$$K(t) = \max_{\bar{\Omega}}\{u(\cdot,t) - w - \lambda t\},$$

we get that K is nonincreasing and since it is also bounded, then there exists some $\bar{K} \in \mathbb{R}$ such that $K(t) \to \bar{K}$ as $t \to +\infty$. Also, using the definition of K we get

$$K(t+t_k) = \max_{\bar{\Omega}}\{v_k(\cdot,t) - w\},$$

and since $\{v_k\}_k$ is uniformly convergent, we let $k \to +\infty$ and we get for each $t \in [0,+\infty)$

$$\bar{K} = \max_{\bar{\Omega}}\{\widetilde{v}(\cdot,t) - w\}.$$

We apply Proposition 5.3.2 and for each $(x,t) \in \bar{\Omega} \times (0,+\infty)$ we get $\widetilde{v}(x,t) = w(x) + \bar{K}$, and then

$$\overline{v} = w + \bar{K} \text{ in } \bar{\Omega}.$$

We deduce

$$v(x,t) \to w + \bar{K} \quad \text{uniformly as } t \to +\infty$$

and by the definition of v

$$||u(\cdot,t) - ct - w - \bar{K}||_\infty = ||v(\cdot,t) - w - \bar{K}||_\infty \to 0 \text{ as } t \to +\infty$$

from which we conclude the result in the case of smooth initial data by replacing w with $w + \bar{K}$. When dealing with $u_0 \in C(\bar{\Omega})$ we use an approximation argument by means of a sequence of smooth initial data u_0^ε such that $u_0^\varepsilon \to u_0$ uniformly in $\bar{\Omega}$ as $\varepsilon \to 0$ (see [29] and [151] for further details).

□

Appendix C Some technical lemmas

In this Appendix we prove some technical lemmas which we use through Chapter 4.

First we prove some lemmas used in the proof of Theorem 4.2.6, that is, the following Lemma 5.3.9, Lemma 4.4.3 and Lemma 5.3.10. In the last Section 5.4 we prove the existence of the blow-up supersolution of Lemma 4.4.7 which we state in Chapter 4, Remark 4.4.6.

We recall that we denote by Hamiltonian of Bellman type an Hamiltonian defined as in (4.11) (see Section 4.2, Chapter 4), satisfying (C),(L), (B1), (B2).

Lemma 5.3.9. *Let H be an Hamiltonian of Bellman type. For all $\hat{s} \in \partial\Omega$, there exists $r = r(\hat{s}) > 0$ and $\gamma, C_2 > 0$ constants such that for all $s \in \bar{B}_r(\hat{s}), \lambda \in \mathbb{R}, p \in \mathbb{R}^n$, it holds:*

(i) if $\hat{s} \in \Gamma_{out}$

$$H(s, p - |\lambda| n(s)) \geq \gamma|\lambda| - C_2|p| - C_2; \tag{5.32}$$

$$H(s, p + |\lambda| n(s)) \leq -\gamma|\lambda| + C_2|p| + C_2; \tag{5.33}$$

(ii) if $\hat{s} \in \Gamma$, then

$$H(s, p - |\lambda| n(s)) \geq \gamma|\lambda| - C_2|p| - C_2; \tag{5.34}$$

$$H(s, p + |\lambda| n(s)) \geq \gamma|\lambda| - C_2|p| - C_2. \tag{5.35}$$

Proof. First we prove (i). Since $\hat{s} \in \Gamma_{\text{out}}$, for $\alpha \in \mathscr{A}$, there exists $r_1, \gamma_1 > 0$ enough small so that

$$b(s, \alpha) \cdot n(s) \geq \gamma_1 \quad \text{for any } s \in \bar{\Omega} \cap B_{r_1}(\hat{s}). \tag{5.36}$$

By the boundedness of b and l and (5.36), we get for some $C_1 > 0$

$$H(s, p - |\lambda| n(s)) \geq -b(s, \alpha) \cdot p + |\lambda| b(s, \alpha) \cdot n(s) - l(s, \alpha) \geq |\lambda|\gamma_1 - C_1|p| - C_1, \tag{5.37}$$

for any $s \in \bar{\Omega} \cap B_{r_1}(\hat{s})$. To prove (5.33), we approximate the supremum in the Hamiltonian by a sequence $\tilde{\alpha} \in \mathscr{A}$. In particular, we take $\tilde{\alpha}$ such that

$$H(s, p + |\lambda| n(s)) \leq -b(s, \tilde{\alpha}) \cdot p - |\lambda| b(s, \tilde{\alpha}) \cdot n(s) - l(s, \tilde{\alpha}) + 1.$$

Since $\hat{s} \in \Gamma_{\text{out}}$ and by using that $\mathscr{A}$ is compact, there exist $r_2, C > 0$ enough small such that

$$b(s, \tilde{\alpha}) \cdot n(s) \geq \gamma_2 \quad \text{for any } s \in \bar{\Omega} \cap B_{r_2}(\hat{s}).$$

Then by the boundedness of b and l we get

$$H(s,p+|\lambda|n(s)) \leq -b(s,\tilde{\alpha})\cdot p-|\lambda|b(s,\tilde{\alpha})\cdot n(s)-l(s,\tilde{\alpha})+1 \leq -\gamma_2|\lambda|+C_1|p|+C_1+1, \tag{5.38}$$

for any $s \in \bar{\Omega}\cap B_{r_2}(\hat{s})$. We conclude the proof of (5.32) and (5.33) by using (5.37) and (5.38) and by denoting $r=\min\{r_1,r_2\}, \gamma=\min\{\gamma_1,\gamma_2\}$ and $C_2=C_1+1$.
Now we prove (ii). Since $\hat{s}\in\Gamma$, there exist $r,\gamma>0$ such that

$$b(s,\alpha_1)\cdot n(s) \geq \gamma \text{ for any } s\in\bar{\Omega}\cap B_r(\hat{s}) \tag{5.39}$$

$$b(s,\alpha_2)\cdot n(s) \leq -\gamma \text{ for any } s\in\bar{\Omega}\cap B_r(\hat{s}) \tag{5.40}$$

. By (5.39), we get

$$H(s,p-|\lambda|n(s)) \geq -b(s,\alpha_1)\cdot p+|\lambda|b(s,\alpha_1)\cdot n(s)-l(s,\alpha_1) \geq |\lambda|\gamma-C_1|p|-C_1,$$

for any $s\in\bar{\Omega}\cap B_r(\hat{s})$, proving (5.34). Analogously, by (5.40), we get

$$H(s,p+|\lambda|n(s)) \geq -b(s,\alpha_2)\cdot p+|\lambda|b(s,\alpha_2)\cdot n(s)-l(s,\alpha_2) \geq |\lambda|\gamma-C_1|p|-C_1,$$

for any $s\in\bar{\Omega}\cap B_r(\hat{s})$ and by denoting $C_2=C_1+1$ we conclude (5.35).

□

Now we prove Lemma 4.4.3, whose statement is given in the proof of Lemma 4.4.1, Chapter 4, Section 4.4.

Proof of Lemma 4.4.3. First we prove (i). Take $-\mathscr{I}_\xi[\phi(\cdot,y)](x)$. By the definition of $\tilde{\phi}$, since χ_ε,ϕ are lipschitz and by (J1) we have

$$\tilde{\phi}(x+j(x,z),y)-\tilde{\phi}(x,y) \leq C\varepsilon^{-1}|z|+C|z|$$

and by (M0) we have

$$\int_{\substack{x+j(x,z)\in\Omega,\\ |z|<\xi}} \tilde{\phi}(x+j(x,z),y)-\tilde{\phi}(x,y)d\mu_x(z) \leq \varepsilon^{-1}C\xi^{1-\sigma}. \tag{5.41}$$

Take now $-\mathscr{I}_\xi[u](x)$. Note that by the boundness of u we have

$$\int_{\substack{x+j(x,z)\in\Omega,\\ |z|>\xi}} u(x+j(x,z))-u(x)d\mu_x(z) \leq 2||u||_\infty \int_{\substack{x+j(x,z)\in\Omega,\\ |z|>\xi}} d\mu_x(z)$$

and then by (M0), we get for some $C > 0$

$$\int_{\substack{x+j(x,z)\in\Omega\\ |z|>\xi}} u(x+j(x,z)) - u(x)d\mu_x(z) \leq C\xi^{-\sigma}. \tag{5.42}$$

Pluggin toghether (5.41) and (5.42), we conclude (i) for some $C_1 > 0$. Similarly we prove (ii).

□

Now we prove the following Lemma, which is used in the proof of Theorem 4.2.6, Chapter 4, Section 4.4.

Lemma 5.3.10. *Let Ω be an open domain of $\mathbb{R}^n$ satisfying (O). Let $R, \nu > 0$ and denote $\psi_R(x) = \psi(R^{-1}|x|)$ where ψ is a smooth function such that*

$$\psi(s) = \begin{cases} 0 & \text{for } 0 \leq s < \frac{1}{2}, \\ \text{increasing} & \text{for } \frac{1}{2} \leq s < 1, \\ ||u||_\infty + ||v||_\infty + 1 & \text{for } s \geq 1. \end{cases} \tag{5.43}$$

Let V be a neighbourhood of the boundary of Ω where the distance from the boundary is smooth and let d be a function which coincides with the signed distance from the boundary in V and is bounded in all the domain. Let $\mathscr{I}$ as in (4.5) *and assume μ satisfies $(M0)$ and j satisfies $(J1)$. Then the function $\phi = \psi_R + \nu d$ satisfies*

$$\mathscr{I}[x](x) \leq o_{\nu,R}(1), \quad |D\phi(x)| \leq o_{\nu,R}(1) \quad \text{for any } x \in \partial\Omega, \tag{5.44}$$

when by $o_{\nu,R}(1)$ we mean that $o_{\nu,R}(1) \to 0$ as $R \to +\infty, \nu \to 0$.

Proof. Note that for any $\xi > 0$ fixed small enough, by the boundedness of ψ and d, we have

$$\mathscr{I}^\xi[\phi](\cdot) \leq o_{\nu,R}(1). \tag{5.45}$$

Now take $x \in \partial\Omega$ and ξ small enough so that $x + j(x,z) \in V$ for any $|z| \leq \xi$. By the definition of ϕ and ψ, since ψ and d are Lipschitz and by (J1), we get

$$\begin{aligned} \phi(x+j(x,z)) - \phi(x) &= \psi_R(x+j(x,z)) - \psi_R(x) + \nu d(x+j(x,z)) - \nu d(x) \\ &= \psi(R^{-1}|x+j(x,z)|) - \psi(R^{-1}|x|) + \nu d(x+j(x,z)) - \nu d(x) \\ &\leq C_j|z|(C(R^{-1}+\nu), \end{aligned} \tag{5.46}$$

where C_j is defined on (J1) and by C we denote the Lipschitz constant of ψ. Then by (5.46) and (M0), we get

$$\mathscr{I}_\xi[\phi](x) \leq \tilde{C}(R^{-1}+\nu)\int_{\mathbb{R}^n} |z|^{1-N-\sigma}dz = o_{\nu,R}(1), \tag{5.47}$$

where $\tilde{C}$ depends on C_j, C and C_μ defined in (M0). Then coupling (5.45) and (5.47) we get

$$\mathscr{I}[\phi](x) \le o_{\nu,R}(1).$$

Note that analogously as above, by writing (5.46) for a general incremenent and using the Lipschitz character of ψ and d on V, we get also

$$|D\phi(x)| \le o_{\nu,R}(1).$$

□

5.4 Blow-up supersolution

In this section we prove Lemma 4.4.7 stated in Chapter 4, and used in the proof of Theorem 4.2.6 (see in particular Lemma 4.4.1, Remark 4.4.6).

We construct the function U_r as showed in the following and then we prove Lemma 5.4.1. Note that Lemma 4.4.7 follows as a consequence of Lemma 5.4.1 and we prove it after the statement of Lemma 5.4.1.

Let $\bar{x} \in \Gamma_{\text{in}}$ and $r = r(\bar{x})$ be given as in assumption (O). We recall that by (O), there exists a $W^{2,\infty}$-diffeomorphism

$$\psi : B_r(\bar{x}) \mapsto \mathbb{R}^n, \tag{5.48}$$

satisfying

$$\psi_n(s) = d(s) \text{ for any } s \in B_r(\bar{x}), \tag{5.49}$$

where d is the signed distance from the boundary of Ω.

We define the blow-up supersolution on $B_r(\bar{x}) \cap \Omega$ as follows

$$U_r(x) = \bar{U}_r(d(x)) \quad \text{ for } x \in B_r(\bar{x}) \cap \Omega, \tag{5.50}$$

where

$$\bar{U}_r = -\log(s) + \frac{3}{2}\log r \quad \text{ if } 0 < s \le r.$$

Note that $\bar{U}_r \in C^\infty(0,r)$ is (nonnegative) monotone and decreasing.

We recall the notation

$$\mathscr{I}_\xi[U_r](x) := P.V. \int_{\substack{x + j(x,z) \in \bar{\Omega}, \\ |z| \le \xi}} [U_(x + j(x,z)) - U_r(x)]\, d\mu_x(z). \tag{5.51}$$

Lemma 5.4.1. *For any $\bar{x} \in \Gamma_{in}$, let $r = r(\bar{x})$ be defined as in assumption (O) and let U_r be defined as in* (5.50). *Then we have for any ξ enough small (with respect to r)*

$$-\mathscr{I}_\xi[U_r](x) \geq -Ad(x)^{-\sigma} \quad in\ B_{\frac{r}{2}}(\bar{x}) \cap \Omega. \tag{5.52}$$

In particular there exists $\tilde{r}(r, A, \sigma) = \tilde{r}$ such that $\tilde{r} \leq r$ and

$$-b(x,\alpha) \cdot DU_r(x) - \mathscr{I}_\xi[U_r](x) \geq 0 \quad in\ B_{\frac{\tilde{r}}{2}}(\bar{x}) \cap \Omega \quad \forall \alpha \in \mathscr{A}. \tag{5.53}$$

Remark 5.4.2. Note that the strict positivity of the drift term on the points of Γ_{in} is essential here to prove (5.53), since the drift term controls the integral term which explodes on the boundary as stated by (5.52).

As a consequence of Lemma 5.4.1, we prove Lemma 4.4.7.

Proof of Lemma 4.4.7. Take $\tilde{r}$ as defined in Lemma 5.4.1 and let $U_{\tilde{r}}$ be defined as in (5.50) for $r = \tilde{r}$. Then $U_{\tilde{r}}$ is a nonnegative decreasing function which trivially satisfies (ii) of Lemma 4.4.7 with $\omega_{\tilde{r}}(s) = \frac{1}{\bar{U}_{\tilde{r}}(s)}$. Moreover, (i) Lemma 4.4.7 follows as a direct application of (5.53) of Lemma 5.4.1. □

Now we prove Lemma 5.4.1.

Proof of Lemma 5.4.1. First we prove (5.52).
Let $\xi \leq C_j^{-1} \frac{r}{2}$, where C_j is defined in (J1). Then, by (J1) and for $|z| \leq \xi$ and $x \in B_{\frac{r}{2}}(\bar{x})$, we have that $x + j(x,z) \in B_r(\bar{x})$. We describe the domain of integration of $\mathscr{I}_\xi[U_r]$ through the diffeomorphism ψ defined in (5.48) as follows

$$x + j(x,z) \in \bar{\Omega} = \psi_n(x + j(x,z)) \geq 0.$$

By the definition of U_r and (5.49), we can write

$$\mathscr{I}_\xi[U_r](x) = -\int_{\substack{\psi_n(x+j(x,z)) \geq 0, \\ |z| \leq \xi}} [\ln(\psi_n(x + j(x,z))) - \ln(\psi_n(x))]\, d\mu_x(z). \tag{5.54}$$

We write

$$\mathscr{I}_\xi[U_r](x) = I^1 + I^2$$

where

$$I^1 = -\int_{\substack{\psi_n(x+j(x,z)) > \psi_n(x), \\ |z| \leq \xi}} [\ln(\psi_n(x + j(x,z))) - \ln(\psi_n(x))]\, d\mu_x(z).$$

and

$$I^2 = -\int_{\substack{\psi_n(x) \geq \psi_n(x+j(x,z)) \geq 0, \\ |z| \leq \xi}} [\ln(\psi_n(x + j(x,z))) - \ln(\psi_n(x))]\, d\mu_x(z).$$

and since

$$I^1 \leq 0$$

we get

$$\mathscr{I}_\xi[U_r](x) \leq -\int_{\substack{\psi_n(x) \geq \psi_n(x+j(x,z)) \geq 0, \\ |z| \leq \xi}} [\ln(\psi_n(x+j(x,z))) - \ln(\psi_n(x))]\, d\mu_x(z). \quad (5.55)$$

We proceed performing a change of variable in order to write the set of integration in terms of $\psi_n(x)$. In other words, we write

$$\psi(x+j(x,z)) - \psi(x) = w \quad (5.56)$$

and then $j(x,z) = \psi^{-1}(\psi(x)+w) - x$. Then by (J0), (J1), (M0) and since ψ is $W^{2,\infty}$, (5.55) becomes

$$\mathscr{I}_\xi[U_r](x) \leq \bar{C} \int_{\substack{0 \geq w_n \geq -\psi_n(x), \\ |w| \leq C\xi}} \left| \ln\left(1 + \frac{w_n}{\psi_n(x)}\right) \right| \frac{dw}{|w|^{n+\sigma}}.$$

for some $\bar{C}, C > 0$. By the change of variable $y = \frac{w}{\psi_n(x)}$, we get

$$\mathscr{I}_\xi[U_r](x) \leq \bar{C}\psi_n(x)^{-\sigma} \int_{0 \geq y_n \geq -1} |\ln(1+y_n)| \frac{dy}{|y|^{n+\sigma}}. \quad (5.57)$$

Note the integral in the right hand side is finite and does not depend on ξ. For convenience of notations we denote

$$A := \bar{C} \int_{0 \geq y_n \geq -1} |\ln(1+y_n)| \frac{dy}{|y|^{n+\sigma}}.$$

Then (5.57) becomes

$$\mathscr{I}_\xi[U_r](x) \leq A d(x)^{-\sigma}, \quad (5.58)$$

which is exactly (5.52).

Now we prove (5.53). First note that, by the definition of U_r, we have

$$DU_r(x) = d(x)^{-1} n \quad \text{in } B_r(\bar{x}) \cap \Omega; \quad (5.59)$$

Then, by (5.52) and (5.59), we have for all $\alpha \in \mathscr{A}$

$$b(x,\alpha) \cdot DU_r(x) + \mathscr{I}_\xi[U_r](x) \leq d(x)^{-1}(b(x,\alpha) \cdot n + d(x)^{1-\sigma} A).$$

Since we are in a neighbourhood of Γ_{in}, $\sigma < 1$ and $\mathscr{A}$ is compact, there exists $0 < \tilde{r} < r$ (depending only on A, σ and r), such that if $x \in B_{\frac{r}{2}}(\bar{x}) \cap \Omega$

$$b(x,\alpha) \cdot DU_r(x) + \mathscr{I}_\xi[U_r](x) \leq 0 \quad \forall \alpha \in \mathscr{A}. \tag{5.60}$$

Then (5.53) follows and we conclude the proof of the Lemma. $\square$

Part III

Quantitative Borell-Brascamp-Lieb inequalities (for power concave functions) and applications to isoperimetric inequalities for some variational functionals

Specific notation of Part III

$V(\cdot)/\lvert\cdot\rvert$	The n-dimensional Lebesgue measure.
$SO(n)$	The group of $n \times n$ orthogonal matrices with values in $\mathbb{R}$ with determinant 1.
ρ	will be a rotation in $SO(n)$.
$\mathscr{S}^n$	The space of $n \times n$ real symmetric matrices.
Ω	will be a bounded open domain of $\mathbb{R}^d$.
$L^p(\Omega)$	The space of functions $f : \Omega \to \mathbb{R}$ such that $\lvert f\rvert^p$ is Lebesgue integrable.
$W^{1,2}(\Omega)$	The Sobolev space of functions in $L^2(\Omega)$ with weak derivative in $L^2(\Omega)$.
$W_0^{1,2}(\Omega)$	The closure of continuously differentiable functions with compact support on Ω, with respect to the topology of $W^{1,2}(\Omega)$.
$I_i, i = 0,1$	see pag 191.
$\Omega_\lambda, \lambda \in (0,1)$	see pag 191.
$\mathscr{M}_q(a,b,\mu)$	see pag 191.
$H(K,L), K,L \subset \mathbb{R}^n$	see pag 193.
$H_0(K,L), K,L \subset \mathbb{R}^n$	see pag 193.
$A(K,L), K,L \subset \mathbb{R}^n$	see pag 194.
$\mathscr{K}_0^n$	see pag 197.
$h_K, K \in \mathscr{K}_0^n$	see pag 197.
$d(K), K \in \mathscr{K}_0^n$	see pag 198.
$w(K), K \in \mathscr{K}_0^n$	see pag 198.
p-concavity	see pag 200.

Chapter 6

Main results and preliminaries

6.1 Introduction and main results

Throughout this part u_0 and u_1 will be real non-negative bounded functions belonging to $L^1(\mathbb{R}^n)$ $(n \geq 1)$ with compact supports Ω_0 and Ω_1 respectively. To avoid triviality, we will assume that

$$I_i = \int_{\mathbb{R}^n} u_i \, dx > 0 \quad \text{for } i = 0, 1\,.$$

For $\lambda \in (0,1)$, denote by Ω_λ the Minkowski convex combination (with coefficient λ) of Ω_0 and Ω_1, that is

$$\Omega_\lambda = (1-\lambda)\Omega_0 + \lambda\Omega_1 = \{(1-\lambda)x_0 + \lambda x_1 \,:\, x_0 \in \Omega_0,\, x_1 \in \Omega_1\}.$$

For $q \in [-\infty, +\infty]$ and $\mu \in (0,1)$ we denote by $\mathscr{M}_q(a,b,\mu)$ the (μ-weighted) *q-mean* of two non-negative numbers a and b, which is defined as follows:

$$\mathscr{M}_q(a,b;\mu) = \begin{cases} \max\{a,b\} & q = +\infty \\ [(1-\mu)a^q + \mu b^q]^{\frac{1}{q}} & \text{if } 0 \neq q \in \mathbb{R} \text{ and } ab > 0 \\ a^{1-\mu} b^{\mu} & \text{if } q = 0 \\ \min\{a,b\} & q = -\infty \\ 0 & \text{when } q \in \mathbb{R} \text{ and } ab = 0\,. \end{cases} \tag{6.1}$$

Note that the arithmetic mean and geometric mean corresponds to the $q = 1$ and $q = 0$, respectively.

Our main results are some refinements of the Borell-Brascamp-Lieb inequality, which we recall in the following theorem.

Theorem 6.1.1 (BBL inequality)**.** *Let* $0 < \lambda < 1, -\frac{1}{n} \leq p \leq \infty$, $0 \leq h \in L^1(\mathbb{R}^n)$ *and assume the following holds*

$$h((1-\lambda)x + \lambda y) \geq \mathscr{M}_p(u_0(x), u_1(y), \lambda) \tag{6.2}$$

for every $x \in \Omega_0$, $y \in \Omega_1$. *Then*

$$\int_{\Omega_\lambda} h(x)\,dx \geq \mathscr{M}_{\frac{p}{np+1}}(I_0, I_1, \lambda)\,. \tag{6.3}$$

Here the number $p/(np+1)$ has to be interpreted in the obvious way in the extremal case, i.e. it is equal to $-\infty$ when $p = -1/n$ and to $1/n$ when $p = \infty$.

The BBL inequality was first proved in a slightly different form for $p > 0$ by Henstock and Macbeath (with $n = 1$) in [106] and by Dinghas in [72]. In its generality it is stated and proved by Brascamp and Lieb in [47] and by Borell in [42] and the equality conditions are discussed in [73]. Since the equality case is rather complicated to state, we refer for the precise statement to [73]. Roughly speaking, equality holds in (15) if and only if the functions $u_i, i = 0, 1, 2$ are almost everywhere equal to some suitable homotheties of the same convex function. The case $p = 0$ was previously proved by Prékopa [138] and Leindler [124] (and rediscovered by Brascamp and Lieb in [46]) and it is usually known as the *Prékopa-Leindler inequality* (PL inequality in the following). It is worth to remark that the PL inequality (and then the BBL inequality, for every p) can be considered as a functional form of the *Brunn-Minkowski inequality*, which in its classical form states that if Ω_1, Ω_0 are two nonempty compact convex sets of $\mathbb{R}^n$ and $\lambda \in (0, 1)$, then

$$|(1-\lambda)\Omega_0 + \lambda\Omega_1| \geq \mathscr{M}_{1/n}(|\Omega_0|, |\Omega_1|, \lambda)$$

and equality holds precisely when Ω_0 and Ω_1 are equal up to translation and dilatation. A generalization to measurable subsets of $\mathbb{R}^n$ has been proved later in [133] and [102]. For more details on the Brunn-Minkowski inequality we refer to Section 6.2 and to [93] as a general and exhaustive reference.

In this part of the thesis we are interested in the investigation of stability problems for the BBL inequality. The typical kind of questions we aim at answering is the following: if $\int_{\Omega_\lambda} h(x)\,dx$ "approximately "coincides with $\mathscr{M}_{\frac{p}{np+1}}(I_0, I_1, \lambda)$, can we infer some kind of "closeness " of the functions u_0, u_1 to the equality condition?

The first step when dealing with a stability problem is to give a precise meaning to the previous question, i.e. choose a measure of the "closeness" of the functions to the equality condition. Depending on the choice of the measure, different kind of results can be obtained.

Significant interest has recently arisen towards the stability of the Brunn-Minkowski inequality and several kind of results have been obtained depending on different measures, see [99, 83, 56, 57, 75, 82]. Concerning the PL inequality, the investigation of stability questions has been recently started by Ball and Böröczky in [12, 13]. Note that all these results are in [48] and all these results are written in terms of the L^1 distance between the involved functions.

Our main achievements are stability results for the BBL inequality in terms of some distance between the support sets Ω_0 and Ω_1 of u_0 and u_1 and some consequent "quantitative" versions of the BBL inequality. With the adjective "quantitative", we mean that we strengthen (6.3) in terms of some distance between the support sets Ω_0 and Ω_1 of u_0 and u_1.

The quantitative versions we give are mainly of two types. The first is written in terms of the Hausdorff distance between (two suitable homothetic copies of) Ω_0 and Ω_1. We recall that the Hausdorff distance $H(K,L)$ between two sets $K,L \subseteq \mathbb{R}^n$ is defined as follows:

$$H(K,L) := \inf\{r \geq 0 : K \subseteq L + r\overline{B}_n,\, K \subseteq L + r\overline{B}_n\}\,,$$

where $B_n = \{x \in \mathbb{R}^n \,:\, |x| < 1\}$ is the (open) unit ball in $\mathbb{R}^n$. Then we set

$$H_0(K,L) = H(\tau_0 K, \tau_1 L), \tag{6.4}$$

where τ_1, τ_0 are two homotheties (i.e. translation plus dilation) such that $|\tau_0 K| = |\tau_1 L| = 1$ and such that the centroids of $\tau_0 K$ and $\tau_1 L$ coincide. By centroid we mean the geometric center, that is the aritmetic mean position of all the points in the shape.

Note that our results apply when u_0 and u_1 are non-negative power concave functions and $p > 0$. We recall that a function $u \geq 0$ is said *p-concave* for some $p \in [-\infty, +\infty]$ if

$$u((1-\lambda)x + \lambda y) \geq \mathscr{M}_p(u(x), u(y); \lambda) \tag{6.5}$$

for all x, $y \in \mathbb{R}^n$ and $\lambda \in (0,1)$ (see Section 6.2 for more details).

Now we are ready to state our first stability result for the BBL inequality.

Theorem 6.1.2. *In the same assumptions and notation of Theorem 6.1.1, assume furthermore that $p > 0$ and*

$$u_0 \text{ and } u_1 \text{ are } p\text{-concave functions} \tag{6.6}$$

(with convex compact supports Ω_0 and Ω_1 respectively). Then, if $H_0(\Omega_0, \Omega_1)$ is small enough, it holds

$$\int_{\Omega_\lambda} h(x)\,dx \geq \mathscr{M}_{\frac{p}{np+1}}(I_0, I_1, \lambda) + \beta\, H_0(\Omega_0, \Omega_1)^{\frac{(n+1)(p+1)}{p}} \tag{6.7}$$

where β is a constant depending only on n, λ, p, I_0, I_1 and the diameters and the measures of Ω_0 and Ω_1.

We provide another quantitative versions of the BBL inequality analogous to (6.7), but in this case the quantitative term depends on te *relative asymmetry* (or *Fraenkel asymmetry*) of

Ω_0 and Ω_1; we recall that the relative asymmetry of two sets K and L is defined as follows

$$A(K,L) := \inf_{x\in\mathbb{R}^n}\left\{\frac{|K\Delta(x+\lambda L)|}{|K|}, \lambda = \left(\frac{|K|}{|L|}\right)^{\frac{1}{n}}\right\}, \tag{6.8}$$

where, for $\Omega \subseteq \mathbb{R}^n$, $|\Omega|$ denotes its Lebesgue measure, while Δ denotes the operation of symmetric difference, i.e. $\Omega\Delta B = (\Omega\setminus B)\cup(B\setminus\Omega)$.

Theorem 6.1.3. *In the same assumptions and notation of Theorem 6.1.2, if $A(\Omega_0,\Omega_1)$ is small enough it holds*

$$\int_{\Omega_\lambda} h(x)\,dx \geq \mathscr{M}_{\frac{p}{np+1}}(I_0,I_1,\lambda) + \delta A(\Omega_0,\Omega_1)^{\frac{2(p+1)}{p}}, \tag{6.9}$$

where δ is a constant depending only on n, λ, p, I_0, I_1 and on the measures of Ω_0 and Ω_1.

The crucial part in the proofs of the above stated results relies on an estimate of the measures of the supports sets of the involved functions; this estimate is contained in Theorem 7.0.1 (Chapter 7), which can be in fact considered our main result. There we prove that if we are close to equality in (6.3), then the measure of $(1-\lambda)\Omega_0+\lambda\Omega_1$ is close to $\mathscr{M}_{1/n}(|\Omega_0|,|\Omega_1|,\lambda)$. Therefore, once Theorem 7.0.1 is proved, the proofs follow by applying different quantitative versions of the classical Brunn-Minkowski inequality (namely [99] and [83], see Chapter 6, Section 6.2). We note also that further recent stability/quantitative results for the Brunn-Minkowski inequality are contained in [56, 57, 75, 82]. A combination of these with Theorem 7.0.1 would lead to further stability/quantitative theorems for the BBL inequality, which could be an interesting topic for further research.

Remark 6.1.4. *(Compact supports)* We consider only compactly supported functions, while BBL inequality holds also when the involved functions are not compactly supported. On the other hand, we are considering only $L^1(\mathbb{R}^n)$ non-negative p-concave functions with $p>0$ (see the next remark for comments about this) and they need to have compact support.

Moreover, even without the restriction of power concavity, this is the only meaningful case of BBL for $p>0$. Indeed, when at least one among u_0 and u_1 has support of infinite measure, the BBL inequality is trivial, since in such a case the left hand side (i.e. $\int h$) must diverge, as it is easily seen: assume $|\Omega_0| = +\infty$ and u_1 does not identically vanish, say there exists x_1 such that $u_1(x_1) = \varepsilon > 0$; then we have $h(x) \geq \lambda^{\frac{1}{p}}\varepsilon$ for $x \in (1-\lambda)\Omega_0 + \lambda x_1$.

Remark 6.1.5. *(p-concavity)* Although all the existing stability results for PL inequality (to our knowledge) are proved assuming some suitable concavity property of the involved functions, see [12, 13, 48], the authors of that papers suggest the possibility that their results may still be valid without such assumptions. And we agree with them. On the other hand, as it can be easily seen, in our results the p-concavity assumption is essential, since they are written in terms of a distance between the support sets of u_0 and u_1. Without such an assumption, one could wildly modify the support sets Ω_0 and Ω_1 (and their distance, whatever you choose) without affecting the L^1 distance between the involved functions.

Remark 6.1.6. *(Explicit constants)* We can provide explicit (but not optimal) estimates for the constants β and δ in Theorem 6.1.2 and Theorem 6.1.3. To this aim and for further use, it is convenient to introduce the following notation:

$$d_i = d(\Omega_i) = \text{ diameter of } \Omega_i\,, \quad \nu_i = |\Omega_i|^{1/n} \text{ for } i = 0,1\,,$$

$$\tilde{d} = \max\left\{\frac{d_0}{\nu_0}, \frac{d_1}{\nu_1}\right\}, \quad M = \max\{\nu_0, \nu_1\}\,, \quad m = \min\{\nu_0, \nu_1\}\,,$$

$$L_i = \max_{\Omega_i} u_i \ \text{ for } i = 0,1, \quad L_\lambda = \mathscr{M}_p(L_0, L_1, \lambda)\,.$$

Then (6.7) holds with

$$\beta = \left[\gamma_n\left(\frac{M}{m}\frac{1}{\sqrt{\lambda(1-\lambda)}} + 2\right)\tilde{d}\right]^{-\frac{(p+1)(n+1)}{p}} \left[2\left(n + \mathscr{M}_{\frac{p}{np+1}}(I_0, I_1; \lambda)^{-1}\right)\right]^{-\frac{p+1}{p}}, \quad (6.10)$$

where

$$\gamma_n = \left(1 + \frac{1}{3\cdot 2^{13}}\right) 3^{\frac{n-1}{n}} 2^{\frac{n+2}{n+1}} n < 6.00025n. \quad (6.11)$$

Similarly, we can observe that (6.9) holds with

$$\delta = \left[\frac{m(1-2^{-1/n})^3}{181^2 n^{13} \Lambda M \left(n + \mathscr{M}_{\frac{p}{np+1}}(I_0, I_1; \lambda)^{-1}\right)}\right]^{\frac{p+1}{p}},$$

where $\Lambda = \max\{\lambda/(1-\lambda), (1-\lambda)/\lambda\}$.

Remark 6.1.7. Theorem 6.1.2 states that (6.7) holds if $H_0(\Omega_0, \Omega_1)$ is *small enough*; this precisely means

$$H_0(\Omega_0, \Omega_1) < (2n)^{-\frac{1}{n+1}} \beta^{-\frac{p}{(n+1)(p+1)}}\,.$$

To avoid this request, we could write (6.7) as follows:

$$\int_{\Omega_\lambda} h(x)\,dx \geq \mathscr{M}_{\frac{p}{np+1}}\left(\int_{\Omega_0} u_0(x)\,dx, \int_{\Omega_1} u_1(x)\,dx, \lambda\right) + \min\left\{B, \beta H_0(\Omega_0, \Omega_1)^{\frac{(n+1)(p+1)}{p}}\right\}$$

where

$$B=\left(\frac{1}{2n}\right)^{\frac{p+1}{p}}. \tag{6.12}$$

A similar remark can be made for Theorem 6.1.3. In particular (6.9) holds when

$$A(\Omega_0,\Omega_1)<(2n)^{-\frac{1}{2}}\delta^{-\frac{p}{2(p+1)}},$$

but we could remove any limitation on the size of $A(\Omega_0,\Omega_1)$ and write

$$\int_{\Omega_\lambda} h(x)\,dx \geq \mathscr{M}_{\frac{p}{np+1}}\left(\int_{\Omega_0} u_0(x)\,dx, \int_{\Omega_1} u_1(x)\,dx, \lambda\right)+\min\left\{B,\, \delta A(\Omega_0,\Omega_1)^{\frac{2(p+1)}{p}}\right\}$$

where B is defined in (6.12).

Remark 6.1.8. *(Dimension sensitivity)* As it is apparent from the previous remarks, the estimates in Theorem 6.1.2 and Theorem 6.1.3 deteriorate quickly as the dimension increases; the same feature is shared by most of the known stability estimates for the Brunn-Minkowski inequality. We notice however that R. Eldan and B. Klartag [75] recently made a new step towards a dimension-sensitive theory for the Brunn-Minkowski inequality, giving rise to the possibility that the stability actually improves as the dimension increases.

Remark 6.1.9. Theorem 6.1.2, Theorem 6.1.3 are written as quantitative forms of the involved inequalities, but they can be obviously interpreted also as stability results for the same inequalities.

6.2 Some preliminaries

Means of non-negative numbers

We have already given the definition of p-mean of two non-negative numbers in (6.1). Here we just recall few useful facts and refer to [103] and [50] for more details. Clearly $\mathscr{M}_p(a,b;\lambda)$ is not-decreasing with respect to a and b for every p and every λ. Moreover a simple consequence of Jensen's inequality is the monotonicity of p-means with respect to p, i.e.

$$\mathscr{M}_p(a,b;\mu)\leq \mathscr{M}_q(a,b;\mu) \quad \text{if } p\leq q. \tag{6.13}$$

We also notice that for every $\mu\in(0,1)$ it holds

$$\lim_{p\to\infty}\mathscr{M}_p(a,b;\mu)=\max\{a,b\} \quad \text{and} \quad \lim_{p\to-\infty}\mathscr{M}_p(a,b;\mu)=\min\{a,b\}.$$

Finally we recall the following technical lemma (for a proof, refer to [93]):

Lemma 6.2.1. *Let* $0<\lambda<1$ *and* a, b, c, d *be nonnegative numbers. If* $p+q>0$, *then*

$$\mathscr{M}_p(a,b,\lambda)\mathscr{M}_q(c,d,\lambda)\geq\mathscr{M}_s(ac,bd,\lambda)$$

where $s=\frac{pq}{p+q}$. The same is true with $s=0$ if $p=q=0$.

Convex bodies and convex functions.

Throughout this part Ω and K, possibly with subscripts, will be bounded convex sets, most often the former open, while the latter a *convex body*, that is a compact convex set with non-empty interior. We denote by $\mathscr{K}_0^n$ the class of convex bodies in $\mathbb{R}^n$.

Next we recall some classical notions of convex geometry, for further details see [147]. Let $L\subset\mathbb{R}^n$ be a convex set, $p\in\mathbb{R}^n\setminus\{0\}$ and $\alpha\in\mathbb{R}$; we set

$$H_{p,\alpha}=\{x\in\mathbb{R}^n\,:\,\langle x,p\rangle=\alpha\}\quad\text{and}\quad H^-_{p,\alpha}=\{x\in\mathbb{R}^n\,:\,\langle x,p\rangle\leq\alpha\}\,.$$

We say that p is an *exterior normal vector* of L at x_0 if $x_0\in L\cap H_{p,\alpha}$ and $L\subseteq H^-_{p,\alpha}$; in such a case, we also say that the hyperplane $H_{p,\alpha}$ is a *support hyperplane* and that $H^-_{p,\alpha}$ is a *supporting halfspace* (with exterior normal vector p) of L.
The *support function* of L is defined in the following way:

$$h(L,x)=\sup\{\langle x,y\rangle:y\in L\},\quad x\in\mathbb{R}^n.$$

If $K\in\mathscr{K}_0^n$, the latter supremum is in fact a maximum and we can write:

$$h(K,x)=\max\{\langle x,y\rangle:y\in K\},\quad x\in\mathbb{R}^n.$$

For any unit vector $\xi\in S^{n-1}$, $h(K,\xi)$ represents the signed distance from the origin of the support plane to K with exterior normal vector ξ. The support function satisfies the following properties:

(i) $h(K,\lambda x)=\lambda h(K,x)\ \forall\lambda\geq 0$.
(ii) $h(K,x+y)\leq h(K,x)+h(K,y)$.

In fact, the latter properties characterize support functions in the following sense: if $f:\mathbb{R}^n\to\mathbb{R}$ is a function that satisfies (i) and (ii), then there is one (and only one) convex body with support function equal to f.

Other useful properties of the support function are the following: let $K,K_1,K_2\in\mathscr{K}_0^n$, then
(iii) $h(K+x_0,\cdot)=h(K,\cdot)+\langle x_0,\cdot\rangle\quad\forall x_0\in\mathbb{R}^n$;
(iv) $h(\lambda K,\cdot)=\lambda h(K,\cdot)\quad\forall\lambda\geq 0$;
(v) $h(K_1+K_2,\cdot)=h(K_1,\cdot)+h(K_2,\cdot)$.
(vi) $h(K_1,\cdot)\leq h(K_2,\cdot)$ if and only if $K_1\subseteq K_2$.

If $K \in \mathscr{K}_0^n$ the number

$$w(K,\xi) = h(K,\xi) + h(K,-\xi), \quad \xi \in S^{n-1}$$

is the *width* of K *in the direction* ξ, that is the distance between the two support hyperplanes of K orthogonal to ξ. The maximum of the width function

$$d(K) = \max\{w(K,\xi) | \xi \in S^{n-1}\}$$

is the *diameter* of K.
The mean width of K is the average of the width of K over all $\xi \in S^{n-1}$, that is

$$w(K) = \frac{1}{n\omega_n} \int_{S^{n-1}} w(K,\xi)\, d\xi = \frac{2}{n\omega_n} \int_{S^{n-1}} h(K,\xi)\, d\xi\,. \tag{6.14}$$

Urysohn's inequality states

$$|K| \leq \omega_n \left(\frac{w(K)}{2} \right)^n, \tag{6.15}$$

equality holding if and only if K is a ball.

The Brunn-Minkowski inequality

As already mentioned in the introduction, the original form of the Brunn–Minkowski inequality involves volumes of convex bodies and states that $V^{1/n}$ is a concave function with respect to Minkowski addition, where $\mathrm{V}(\cdot)$ denotes the n-dimensional Lebesgue measure and the Minkowski addition of convex sets is defined as follows:

$$A + B = \{x + y \mid x \in A,\ y \in B\}$$

In particular, let $\lambda \in [0,1]$ and let Ω_0 and Ω_1 be convex subsets of $\mathbb{R}^n$; we define their *Minkowski linear combination* Ω_λ as

$$\Omega_\lambda = (1-\lambda)\Omega_0 + \lambda\Omega_1 = \{(1-\lambda)x_0 + \lambda x_1 \,:\, x_i \in \Omega_i\,, i = 0,1\}\,. \tag{6.16}$$

With this notation, the classical Brunn-Minkowski inequality reads

$$\mathrm{V}(K_\lambda)^{\frac{1}{n}} \geq (1-\lambda)\,\mathrm{V}(K_0)^{\frac{1}{n}} + \lambda\,\mathrm{V}(K_1)^{\frac{1}{n}} \tag{6.17}$$

for $K_0, K_1 \in \mathscr{K}_0^n$ and $\lambda \in [0,1]$ and it can be also written in the following equivalent multiplicative form

$$V(K_\lambda) \geq V(K_0)^{1-\lambda} V(K_1)^{\lambda}\,.$$

As it is well known, the Brunn-Minkowski inequality and the PL inequality are equivalent (notice that the way from the latter to the former is almost straightforward by taking $u_0 = \chi_{K_0}$, $u_1 = \chi_{K_1}$ and $h = \chi_{K_\lambda}$, where χ_A represents the characteristic function of the set A).

Inequality (6.17) is one of the fundamental results in the theory of convex bodies and several other important inequalities, e.g. the isoperimetric inequality, can be deduced from it. It can be extended to measurable sets and it holds also, with the right exponents, for the other quermassintegrals. We refer the interested reader to [147] and to the survey paper [93] for this topic; see also [119, 63]. It is also interesting to notice that analogues of (6.17) hold for many variational functionals, see for instance [43, 44, 40, 47, 59–61, 104, 143, 144].

We recall two quantitative versions of (6.17) which will be used later.

The first proposition is due to Groemer [99].

Proposition 6.2.2. *Let $K_0, K_1 \in \mathscr{K}_0^n$, $n \geq 2$, $\lambda \in (0,1)$ and let*

$$K_\lambda = (1-\lambda)K_0 + \lambda K_1.$$

Set $\nu_i = |K_i|^{\frac{1}{n}}$. Let $\tilde{d} = \max\{\frac{d(K_0)}{\nu_0}; \frac{d(K_1)}{\nu_1}\}$ and $M = \max\{\nu_0, \nu_1\}, m = \min\{\nu_0, \nu_1\}$. Then

$$|K_\lambda| \geq \mathscr{M}_{\frac{1}{n}}(|K_0|, |K_1|, \lambda)\left(1 + \omega H_0(K_0, K_1)^{(n+1)}\right)$$

where

$$\omega = \left(\gamma_n\left(\frac{M}{m}\frac{1}{\sqrt{\lambda(1-\lambda)}} + 2\right)\tilde{d}\right)^{-(n+1)},$$

H_0 is defined as in (6.4) *and*

$$\gamma_n = (1 + \frac{1}{3}2^{-13})3^{\frac{n-1}{n}}2^{\frac{n+2}{n+1}}n < 6.00025n.$$

The second proposition is due to Figalli, Maggi, Pratelli [83, 84].

Proposition 6.2.3. *Let $K_0, K_1 \in \mathscr{K}_0^n$, $\lambda \in (0,1)$ and let*

$$K_\lambda = (1-\lambda)K_0 + \lambda K_1.$$

Then

$$|K_\lambda| \geq \mathscr{M}_{\frac{1}{n}}(|K_0|, |K_1|, \lambda)\left(1 + \frac{nm}{\Lambda M}\left(\frac{A(K_0, K_1)}{\theta_n}\right)^2\right),$$

where $A(K_0, K_1)$ is defined in (6.8)*, m and M are defined as in the previous theorem, $\Lambda = \max\{\lambda/(1-\lambda), (1-\lambda)/\lambda\}$ and θ_n is a constant depending on n with polynomial growth.*

In particular

$$\theta_n \leq \frac{362n^7}{(2-2^{\frac{n-1}{n}})^{\frac{3}{2}}}.$$

We further recall that a very recent stability result for the Brunn-Minkowski inequality by Figalli and Jerison is contained in [82] and previous results have been obtained by M. Christ in [56, 57].

Power concave functions

Let Ω be a convex set in $\mathbb{R}^n$ and $p \in [-\infty, \infty]$. A nonnegative function u defined in Ω is said *p-concave* if

$$u((1-\lambda)x+\lambda y) \geq \mathscr{M}_p(u(x), u(y); \lambda)$$

for all x, $y \in \Omega$ and $\lambda \in (0,1)$. In the cases $p=0$ and $p=-\infty$, u is also said log-concave and quasi-concave in Ω, respectively.

In other words, a non-negative function u, with convex support Ω, is p-concave if:
- it is a non-negative constant in Ω, for $p=+\infty$;
- u^p is concave in Ω, for $p>0$;
- $\log u$ is concave in Ω, for $p=0$;
- u^p is convex in Ω, for $p<0$;
- it is quasi-concave, i.e. all of its superlevel sets are convex, for $p=-\infty$.

Notice that $p=1$ corresponds to usual concavity. Notice also that from (6.13) it follows that if u is p-concave, then u is q-concave for every $q \leq p$ (this in particular means that quasi-concavity is the weakest concavity property one can imagine).

The solutions of elliptic Dirichlet problems in convex domains are often power concave. Two famous results state for instance that the first positive eigenfunction of the Laplace operator in a convex domain is log-concave [46] and that the square root of the solution to the torsion problem in a convex domain is concave [117, 118, 123]. For recent results and updated references (in the elliptic and parabolic cases), see for instance [39, 109].

The concavity properties of a function u can be expressed in terms of its level sets. Precisely it is easily seen that a function u is concave if and only if

$$\{u \geq (1-\lambda)t_0+\lambda t_1\} \supseteq (1-\lambda)\{u \geq t_0\}+\lambda\{u \geq t_1\}$$

for every $t_0, t_1 \in \mathbb{R}$ and every $\lambda \in (0,1)$.

More generally, we have the following characterization of power concave functions, which easily follows from the above property.

Proposition 6.2.4. *A non-negative function u is p-concave in a convex domain Ω for some $p \in [-\infty, +\infty)$ if and only if*

$$\{x \in \Omega : u(x) \geq \mathscr{M}_p(t_0, t_1, \lambda)\} \supseteq (1-\lambda)\{x \in \Omega : u(x) \geq t_0\} + \lambda\{x \in \Omega : u(x) \geq t_1\}$$

for every $t_0, t_1 \geq 0$ and every $\lambda \in (0,1)$.

Let μ be the distribution function of u, i.e.

$$\mu(t) = |\{u \geq t\}|. \tag{6.18}$$

Then, as a direct consequence of the Brunn-Minkowski inequality and Proposition 6.2.4, we have the following.

Proposition 6.2.5. *If u is p-concave for some $p \neq 0$, then*

$$\mu(t^{1/p})^{1/n} \text{ is concave in } t\,.$$

If u is log-concave (corresponding to $p = 0$), then

$$\mu(e^t)^{1/n} \text{ is concave in } t\,.$$

The (p,λ)-convolution of non-negative functions

Let $p \in \mathbb{R}$, $\mu \in (0,1)$, and u_0, u_1 non-negative functions with compact convex support Ω_0 and Ω_1, as usual in this paper.

The (p,λ)-*convolution* of u_0 and u_1 (also called p-Minkowski sum, see [119]) is the function defined as follows:

$$\begin{aligned} u_{p,\lambda}(x) = \sup\big\{\, M_p\big(u_0(x_0), u_1(x_1); \lambda\big) \,:\\ x = (1-\lambda)x_0 + \lambda x_1\,,\, x_i \in \overline{\Omega_i},\, i = 0,1\big\}. \end{aligned} \tag{6.19}$$

The above definition can be extended to the case $p = \pm\infty$, but we do not need here. Notice that (6.13) yields

$$u_{q,\lambda} \leq u_{p,\lambda} \qquad \text{if } q \leq p\,. \tag{6.20}$$

It is easily seen that the support of $u_{p,\lambda}$ is $\Omega_\lambda = (1-\lambda)\Omega_0 + \lambda\Omega_1$, and that the continuity of u_0 and u_1 yields the continuity of $u_{p,\lambda}$, in particular if $u_i \in C(\overline{\Omega_i})$ for $i = 0,1$, then $u_{p,\lambda} \in C(\overline{\Omega_\lambda})$.

Let $p \neq 0$; then, roughly speaking, the graph of $u^p_{p,\lambda}$ is obtained as the Minkowski convex combination (with coefficient λ) of the hypographs of u_0^p and u_1^p; precisely we have

$$K_\lambda^{(p)} = (1-\lambda)K_0^{(p)} + \lambda K_1^{(p)}\,,$$

where

$$K_\lambda^{(p)} = \{(x,t) \in \mathbb{R}^{n+1} \,:\, x \in \Omega_\lambda\,, 0 \le t \le u_{p,\lambda}(x)^p\}\,, \tag{6.21}$$

$$K_i^{(p)} = \{(x,t) \in \mathbb{R}^{n+1} \,:\, x \in \Omega_i, 0 \le t \le u_i(x)^p\}\,, \quad i = 0,1\,. \tag{6.22}$$

In other words, the (p,λ)-convolution of u_0 and u_1 corresponds to the $(1/p)$-power of the supremal convolution (with coefficient λ) of u_0^p and u_1^p. When $p = 0$, the above geometric considerations continue to hold with logarithm in place of power p and exponential in place of power $1/p$. When $p = 1$, $u_{1,\lambda}$ is just the usual supremal convolution of u_0 and u_1 (see for instance [143, §3]). For more details on infimal/supremal convolutions of convex/concave functions, see [139, 149].

From the definition of $u_{p,\lambda}$ and the monotonicity of p-means with respect to p, we get

$$u_{p,\lambda} \le u_{q,\lambda} \quad \text{for } -\infty \le p \le q \le +\infty\,. \tag{6.23}$$

Chapter 7

Quantitative BBL inequalities

The main results of this Chapter are Theorem 6.1.2 and Theorem 6.1.3. They essentially stem from the following stability result for the BBL inequality, which can be considered the main result and we will prove in Section 7.2.

Theorem 7.0.1. *In the same assumptions and notation of Theorem 6.1.2 and Theorem 6.1.3 and Remark 6.1.6 of Section 6.1, if for some (small enough) $\varepsilon > 0$ it holds*

$$\int_{\Omega_\lambda} h(x)\,dx \le \mathscr{M}_{\frac{p}{np+1}}\left(\int_{\Omega_0} u_0(x)\,dx, \int_{\Omega_1} u_1(x)\,dx;\lambda\right) + \varepsilon, \tag{7.1}$$

then

$$|\Omega_\lambda| \le \mathscr{M}_{\frac{1}{n}}(|\Omega_0|, |\Omega_1|, \lambda)\left[1 + \eta\,\varepsilon^{\frac{p}{p+1}}\right]. \tag{7.2}$$

where

$$\eta \le 2\left(n + \mathscr{M}_{\frac{p}{np+1}}\Big(\int_{\Omega_0} u_0(x)\,dx, \int_{\Omega_1} u_1(x)\,dx;\lambda\Big)^{-1}\right). \tag{7.3}$$

Remark 7.0.2. "Small enough" (referred to ε in the statement of Theorem 7.0.1) precisely means

$$\varepsilon \le \left(\frac{1}{2n}\right)^{\frac{p+1}{p}}$$

and we could make similar comments as in Remark 6.1.6 and Remark 6.1.7 of Section 6.1. This number depends on n (and tends to 0 as $n \to \infty$), then the result of Theorem 7.0.1 is dimension sensitive (see Remark 6.1.8).

Remark 7.0.3. Looking at the proof of Theorem 7.0.1, one can understand that the same argument can be applied to any level set of the involved functions. Then we could possibly write stability results for the BBL inequality in terms of some L^q distance of u_0 and u_1. On the other hand, for applications as the ones we present in Chapter 8, it is natural to consider some distance between the supports better than some distance between the functions.

Now we prove Theorem 6.1.2

Proof. We argue by contradiction. Suppose that

$$\int_{\Omega_\lambda} h(x)\,dx < \mathscr{M}_{\frac{p}{np+1}}\left(\int_{\Omega_0} u_0(x)\,dx, \int_{\Omega_1} u_1(x)\,dx, \lambda\right) + \beta H_0(\Omega_0,\Omega_1)^{\frac{(n+1)(p+1)}{p}}$$

where β is defined in Remark 6.1.6 of Section 6.1, see (6.10). For convenience of the reder we recall the definition

$$\beta = \left[\gamma_n\left(\frac{M}{m}\frac{1}{\sqrt{\lambda(1-\lambda)}}+2\right)\tilde{d}\right]^{-\frac{(p+1)(n+1)}{p}} \left[2\left(n+\mathscr{M}_{\frac{p}{np+1}}(I_0,I_1;\lambda)^{-1}\right)\right]^{-\frac{p+1}{p}},$$

where

$$d_i = d(\Omega_i) = \text{ diameter of } \Omega_i\,, \quad \nu_i = |\Omega_i|^{1/n} \text{ for } i=0,1\,,$$

$$\tilde{d} = \max\left\{\frac{d_0}{\nu_0},\frac{d_1}{\nu_1}\right\}, \quad M = \max\{\nu_0,\nu_1\}\,, \quad m = \min\{\nu_0,\nu_1\}\,,$$

and

$$\gamma_n = \left(1+\frac{1}{3\cdot 2^{13}}\right)3^{\frac{n-1}{n}}2^{\frac{n+2}{n+1}}n < 6.00025n \tag{7.4}$$

Then we apply Theorem 7.0.1 and we get

$$|\Omega_\lambda| < \mathscr{M}_{\frac{1}{n}}(|\Omega_0|,|\Omega_1|,\lambda)\left(1+\eta\beta^{\frac{p}{p+1}}H_0(\Omega_0,\Omega_1)^{n+1}\right),$$

where η is like in (7.3). Then we use Proposition 6.2.2 and, thanks to the definition of the constant β, we easily get a contradiction. □

Regarding Theorem 6.1.3, we notice that it can be proved precisely in the same way, using the quantitative version of the Brunn-Minkowski inequality proved by Figalli, Maggi and Pratelli, that is Proposition 6.2.3, in place of Proposition 6.2.2.

7.1 An alternative proof of the BBL inequality

Before giving the proof of Theorem 7.0.1, we recall here an alternative proof of Theorem 6.1.1 for power concave functions. The argument will be useful for the proof of Theorem 7.0.1.

Proof. First of all, we define $u_{p,\lambda}$ as in (6.19), i.e.

$$u_{p,\lambda}(x) = \sup\big\{ \, M_p\big(u_0(x_0), u_1(x_1); \lambda\big) \; : \\ x = (1-\lambda)x_0 + \lambda x_1 \,, \, x_i \in \overline{\Omega_i}, \, i = 0, 1 \big\}.$$

Notice that assumption (6.2) of Theorem 6.1.1 implies

$$h \geq u_{p,\lambda} \quad \text{in } \mathbb{R}^n .$$

Let

$$I_i = \int_{\Omega_i} u_i \, dx \quad i = 0, 1 \,,$$

and

$$I_\lambda = \int_{\Omega_\lambda} u_{p,\lambda} \, dx \,.$$

As declared at the beginning, we assume

$$I_i > 0 \; i = 0, 1.$$

and

$$L_i = \max_{\Omega_i} u_i < \infty \quad i = 0, 1.$$

Notice that the very definition of $u_{p,\lambda}$ yields

$$L_\lambda = \max_{\Omega_\lambda} u_{p,\lambda} = \mathscr{M}_p(L_0, L_1, \lambda).$$

Let

$$\mu_i(s) = |\{u_i \geq s\}| \quad i = 0, 1 \,, \qquad \mu_\lambda(s) = |\{u_{p,\lambda} \geq s\}|$$

(notice that the distribution functions μ_0, μ_1 and μ_λ are continuous thanks to the p-concavity of the involved functions). Then

$$I_i = \int_0^{L_i} \mu_i(s) \, ds \quad i = 0, 1, \lambda.$$

The definition of u_λ yields

$$\{u_{p,\lambda} \geq \mathscr{M}_p(s_0, s_1; \lambda)\} \supseteq (1-\lambda)\{u_0 \geq s_0\} + \lambda\{u_1 \geq s_1\}$$

for $s_0 \in [0, L_0]$, $s_1 \in [0, L_1]$. Then, using the Brunn-Minkowski inequality (6.17) with $K_0 = \{u_0 \geq s_0\}$, $K_1 = \{u_1 \geq s_1\}$, we get

$$|(1-\lambda)\{u_0 \geq s_0\} + \lambda\{u_1 \geq s_1\}| \geq \mathscr{M}_{\frac{1}{n}}(\mu_0(s_0), \mu_1(s_1), \lambda)$$

and then

$$\mu_\lambda(\mathscr{M}_p(s_0,s_1;\lambda)) \geq \mathscr{M}_{\frac{1}{n}}(\mu_0(s_0),\mu_1(s_1),\lambda). \tag{7.5}$$

Define the functions $s_i : [0,1] \to [0,L_i]$ for $i=0,1$ such that

$$s_i(t) : \frac{1}{I_i}\int_0^{s_i(t)} \mu_i(s)\,ds = t \quad \text{for } t \in [0,1]. \tag{7.6}$$

Notice that s_i is strictly increasing, then it is differentiable almost everywhere and differentiating (7.6) we obtain

$$\frac{s_i'(t)\mu_i(s_i(t))}{I_i} = 1 \text{ a.e. } t \in [0,1], \quad i=0,1. \tag{7.7}$$

It is also easily seen that s_i is continuous and by (7.7) its derivative s_i' coincides almost everywhere with a continuous function in $[0,1)$; hence, as a derivative, in fact it is continuous in the whole $[0,1)$ and finally $s_i \in C^1([0,1))$. Moreover, since μ_i is decreasing and s_i is increasing, by (7.7) we can also see that s_i' is increasing, which yields s_i is convex in $[0,1]$.

Now set

$$s_\lambda(t) = \mathscr{M}_p(s_0(t),s_1(t),\lambda) \quad t \in [0,1]$$

and calculate

$$s_\lambda'(t) = ((1-\lambda)s_0'(t)s_0(t)^{p-1} + \lambda s_1'(t)s_1(t)^{p-1})s_\lambda(t)^{1-p} \quad \text{a.e. } t \in [0,1]. \tag{7.8}$$

Notice that the map $s_\lambda : [0,1] \mapsto [0,L_\lambda]$ is strictly increasing, then invertible; let us denote by $t_\lambda : [0,L_\lambda] \mapsto [0,1]$ its inverse map.

Then

$$\begin{aligned} I_\lambda &= \int_0^{L_\lambda} \mu_\lambda(s)\,ds = \int_0^1 \mu_\lambda(s_\lambda(t))s_\lambda'(t)\,dt \\ &= \int_0^1 \mu_\lambda(s_\lambda(t))\mathscr{M}_1(s_0'(t)s_0(t)^{p-1}, s_1'(t)s_1(t)^{p-1})s_\lambda(t)^{1-p}\,dt. \end{aligned} \tag{7.9}$$

Thanks to (7.5), we get

$$\mu_\lambda(s_\lambda(t)) \geq \mathscr{M}_{\frac{1}{n}}(\mu_0(s_0(t)),\mu_1(s_1(t)),\lambda) \quad t \in [0,1] \tag{7.10}$$

and coupling (7.9) and (7.10) we arrive to

$$I_\lambda \geq \int_0^1 \mathscr{M}_{\frac{1}{n}}(\mu_0(s_0(t)),\mu_1(s_1(t)),\lambda)\,\mathscr{M}_1(s_0'(t)s_0(t)^{p-1}, s_1'(t)s_1(t)^{p-1},\lambda)\,s_\lambda(t)^{1-p}\,dt. \tag{7.11}$$

Next we use Lemma 6.2.1 with $p = \frac{1}{n}$ and $q = 1$ to obtain

$$\mathscr{M}_{\frac{1}{n}}(\mu_0(s_0), \mu_1(s_1), \lambda)\mathscr{M}_1(s_0' s_0^{p-1}, s_1' s_1^{p-1}, \lambda) \geq \mathscr{M}_{\frac{1}{n+1}}(\mu_0(s_0)s_0^{p-1}s_0', \mu_1(s_1)s_1^{p-1}s_1', \lambda)$$

for $s_0 \in [0, L_0]$, $s_1 \in [0, L_1]$. Then (7.11) yields

$$I_\lambda \geq \int_0^1 \mathscr{M}_{\frac{1}{n+1}}(\mu_0(s_0(t))s_0(t)^{p-1}s_0'(t), \mu_1(s_1(t))s_1(t)^{p-1}s_1'(t), \lambda)s_\lambda(t)^{1-p}\,dt. \tag{7.12}$$

Since

$$s_\lambda^{1-p} = \mathscr{M}_p(s_0, s_1, \lambda)^{1-p} = \mathscr{M}_{\frac{p}{1-p}}(s_0^{1-p}, s_1^{1-p}, \lambda), \tag{7.13}$$

using again Lemma 6.2.1 with $p = \frac{1}{n+1}$ and $q = \frac{p}{1-p}$ we get

$$\begin{aligned} \mathscr{M}_{\frac{1}{n+1}}(\mu_0(s_0)s_0^{p-1}s_0', \mu_1(s_1)s_1^{p-1}s_1', \lambda)\mathscr{M}_{\frac{p}{1-p}}(s_0^{1-p}, s_1^{1-p}, \lambda)& \\ \geq \mathscr{M}_{\frac{p}{np+1}}(\mu_0(s_0)s_0', \mu_1(s_1)s_1', \lambda).& \end{aligned} \tag{7.14}$$

Then coupling (7.14) with (7.12) we obtain

$$I_\lambda \geq \int_0^1 \mathscr{M}_{\frac{p}{np+1}}(\mu_0(s_0(t))s_0'(t), \mu_1(s_1(t))s_1'(t), \lambda)\,dt\,,$$

whence, thanks to (7.7), we finally arrive to

$$I_\lambda \geq \int_0^1 \mathscr{M}_{\frac{p}{np+1}}(I_0, I_1, \lambda)\,dt = \mathscr{M}_{\frac{p}{np+1}}(I_0, I_1, \lambda)$$

This concludes the proof. $\square$

7.2 The main stability result

We prove Theorem 7.0.1.

Proof. First of all notice that Brunn-Minkowsi inequality states

$$|\Omega_\lambda| \geq \mathscr{M}_{\frac{1}{n}}(|\Omega_0|, |\Omega_1|, \lambda)\,,$$

and if equality holds, there is nothing to prove. Then let us assume

$$|\Omega_\lambda| = \mathscr{M}_{\frac{1}{n}}(|\Omega_0|, |\Omega_1|, \lambda) + \tau, \tag{7.15}$$

for some $\tau > 0$. Our aim is to find and estimate on τ depending on ε, that is $\tau < f(\varepsilon)$ (with $\lim_{\varepsilon \to 0} f(\varepsilon) = 0$).

We use the same notation as in the proof of Theorem 6.1.1 given in the previous section and following the same argument we arrive again to (7.5) and then (7.10).

Now, given any $\delta > 0$, set

$$F_\delta = \{t \in [0,1] \,:\, \mu_\lambda(s_\lambda(t)) > \mathscr{M}_{\frac{1}{n}}(\mu_0(s_0(t)), \mu_1(s_1(t)), \lambda) + \delta\} \tag{7.16}$$

and

$$\Gamma_\delta = \{s_\lambda(t) \,:\, t \in F_\delta\}. \tag{7.17}$$

Notice that F_δ and Γ_δ are measurable sets, thanks to Proposition 6.2.5 and to the monotonicity and regularity of the s_i's.

Then we have

$$\begin{aligned}
I_\lambda &= \int_0^{L_\lambda} \mu_\lambda(s)\,ds = \int_0^1 \mu_\lambda(s_\lambda(t))s'_\lambda(t)\,dt \\
&= \int_{F_\delta} \mu_\lambda(s_\lambda(t))s'_\lambda(t)\,dt + \int_{[0,1]\setminus F_\delta} \mu_\lambda(s_\lambda(t))s'_\lambda(t)\,dt \\
&\geq \int_{F_\delta} \left[\mathscr{M}_{\frac{1}{n}}(\mu_0(s_0(t)), \mu_1(s_1(t)), \lambda) + \delta\right] s'_\lambda(t)\,dt + \int_{[0,1]\setminus F_\delta} \mu_\lambda(s_\lambda(t))s'_\lambda(t)\,dt \\
&\geq \int_0^1 \mathscr{M}_{\frac{1}{n}}(\mu_0(s_0(t)), \mu_1(s_1(t)), \lambda)s'_\lambda(t)\,dt + \delta \int_{F_\delta} s'_\lambda(t)\,dt \\
&= \int_0^1 \mathscr{M}_{\frac{1}{n}}(\mu_0(s_0(t)), \mu_1(s_1(t)), \lambda)s'_\lambda(t)\,dt + \delta\,|\Gamma_\delta|
\end{aligned}$$

where in the first inequality we have used the definition of F_δ, in the second we have used (7.10) and in the last equality we have used the definition of Γ_δ (and the change of variable $s = s_\lambda(t)$).

Continuing to argue as in the proof of Theorem 6.1.1 given in the previous section, we find

$$\int_0^1 \mathscr{M}_{\frac{1}{n}}(\mu_0(s_0(t)), \mu_1(s_1(t)), \lambda)s'_\lambda(t)\,dt \geq \mathscr{M}_{\frac{p}{np+1}}(I_0, I_1, \lambda).$$

Moreover from (7.1) we know that

$$\mathscr{M}_{\frac{p}{np+1}}(I_0, I_1, \lambda) + \varepsilon \geq I_\lambda$$

and so we can conclude

$$\mathscr{M}_{\frac{p}{np+1}}(I_0, I_1, \lambda) + \varepsilon \geq I_\lambda \geq \mathscr{M}_{\frac{p}{np+1}}(I_0, I_1, \lambda) + \delta\,|\Gamma_\delta|$$

which implies that

$$|\Gamma_\delta| \leq \varepsilon/\delta. \tag{7.18}$$

Take now

$$\delta = \varepsilon^{\alpha}/L_{\lambda}$$

for some $0 < \alpha < 1$. Then (7.18) reads

$$|\Gamma_{\varepsilon^{\alpha}/L_{\lambda}}| < \varepsilon^{1-\alpha} L_{\lambda} \,. \tag{7.19}$$

Let u_{λ} be defined in (6.19). Then, thanks to assumption (6.6), u_{λ} is p-concave, that is the following inclusion holds

$$\{z : u_{\lambda}(z) \geq \mathscr{M}_p(\ell_0, \ell_1, \xi)\} \supseteq (1-\xi)\{x : u_{\lambda}(x) \geq \ell_0\} + \xi\{y : u_{\lambda}(y) \geq \ell_1\}. \tag{7.20}$$

for $\xi \in [0,1]$, $\ell_0 \in [0, L_0]$ and $\ell_1 \in [0, L_1]$.
Let us choose

$$\ell_0 = 0, \quad \ell_1 = L_{\lambda}. \tag{7.21}$$

By (7.19), we can find $\bar{t} > 0$ such that

$$s_{\lambda}(\bar{t}) \leq \varepsilon^{1-\alpha} L_{\lambda} \,, \tag{7.22}$$

and

$$\mu_{\lambda}(s_{\lambda}(\bar{t})) \leq \mathscr{M}_{\frac{1}{n}}(\mu_0(s_0(\bar{t})), \mu_1(s_1(\bar{t})), \lambda) + \varepsilon^{\alpha} L_{\lambda}^{-1}. \tag{7.23}$$

Let

$$\xi = \left(\frac{s_{\lambda}(\bar{t})}{L_{\lambda}}\right)^{p}. \tag{7.24}$$

From (7.22) we have

$$\xi \leq \varepsilon^{(1-\alpha)p}. \tag{7.25}$$

With these choices of ℓ_0, ℓ_1 and ξ, we have $s_{\lambda}(\bar{t}) = \mathscr{M}_p(\ell_0, \ell_1, \xi)$ and (7.20) reads

$$\{u_{\lambda} \geq s_{\lambda}(\bar{t})\} \supseteq (1-\xi)\Omega_{\lambda} + \xi\{u_{\lambda} \geq L_{\lambda}\}$$

From the Brunn-Minkowski inequality we get

$$|\{u_{\lambda} \geq s_{\lambda}(\bar{t})\}| \geq \left((1-\xi)|\Omega_{\lambda}|^{\frac{1}{n}} + \xi\,|\{u_{\lambda} \geq L_{\lambda}\}|^{\frac{1}{n}}\right)^{n}.$$

Using (7.15) and neglecting $|\{u_{\lambda} \geq L_{\lambda}\}|$ (notice that $\{u_{\lambda} \geq L_{\lambda}\} = (1-\lambda)\{u_0 \geq L_0\} + \lambda\{u_1 \geq L_1\}$ and, if the involved functions are strictly p-concave, as we can assume without loss of generality, these three sets reduce to a single point, then they all have zero measure), we get

$$|\{u_{\lambda} \geq s_{\lambda}(\bar{t})\}| \geq (1-\xi)^n \mathscr{M}_{\frac{1}{n}}(|\Omega_0|, |\Omega_1|, \lambda) + (1-\xi)^n \tau.$$

Then, by (7.23) we have

$$\varepsilon^{\alpha} L_{\lambda}^{-1} + \mathscr{M}_{\frac{1}{n}}(\mu_0(s_0(\bar{t})), \mu_1(s_1(\bar{t})), \lambda) \geq (1-\xi)^n \mathscr{M}_{\frac{1}{n}}(|\Omega_0|, |\Omega_1|, \lambda) + (1-\xi)^n \tau .$$

Since $\mu_0(s_0(\bar{t})) \leq |\Omega_0|$ and $\mu_1(s_1(\bar{t})) \leq |\Omega_1|$ and thanks to the monotonicity of the mean $\mathscr{M}_{\frac{1}{n}}$, the previous formula implies

$$\varepsilon^{\alpha} L_{\lambda}^{-1} + \mathscr{M}_{\frac{1}{n}}(|\Omega_0|, |\Omega_1|, \lambda) \geq (1-\xi)^n \mathscr{M}_{\frac{1}{n}}(|\Omega_0|, |\Omega_1|, \lambda) + (1-\xi)^n \tau ,$$

whence

$$\tau \leq \left(\varepsilon^{\alpha} L_{\lambda}^{-1} + \mathscr{M}_{\frac{1}{n}}(|\Omega_0|, |\Omega_1|, \lambda)[1-(1-\xi)^n] \right) (1-\xi)^{-n}.$$

Since $(1-\xi)^n \geq 1 - n\xi \geq 1/2$ for $0 \leq \xi \leq \frac{1}{2n}$, we get

$$\tau \leq 2 \left(\varepsilon^{\alpha} L_{\lambda}^{-1} + n \, \mathscr{M}_{\frac{1}{n}}(|\Omega_0|, |\Omega_1|, \lambda) \, \xi \right) . \tag{7.26}$$

Take $\alpha = \frac{p}{p+1}$ and ε small enough (precisely $\varepsilon \leq (\frac{1}{2n})^{\frac{(p+1)}{p}}$), then (7.26) reads

$$|\Omega_{\lambda}| \leq \mathscr{M}_{\frac{1}{n}}(|\Omega_0|, |\Omega_1|, \lambda) + 2 \left(L_{\lambda}^{-1} + n \mathscr{M}_{\frac{1}{n}}(|\Omega_0|, |\Omega_1|, \lambda) \right) \varepsilon^{\frac{p}{p+1}} . \tag{7.27}$$

Since clearly $I_i \leq L_i |\Omega_i|$, for $i = 0, 1, \lambda$, we get

$$L_{\lambda} \geq \mathscr{M}_p(I_0/|\Omega_0|, I_1/|\Omega_1|; \lambda)$$

and Lemma 6.2.1 implies

$$L_{\lambda} \geq \frac{\mathscr{M}_{\frac{p}{np+1}}(I_0, I_1; \lambda)}{\mathscr{M}_{\frac{1}{n}}(|\Omega_0|, |\Omega_1|; \lambda)} .$$

Combining the latter with (7.27) we obtain

$$|\Omega_{\lambda}| \leq \mathscr{M}_{\frac{1}{n}}(|\Omega_0|, |\Omega_1|, \lambda) \left[1 + 2 \left(n + \mathscr{M}_{\frac{p}{np+1}}(I_0, I_1; \lambda)^{-1} \right) \varepsilon^{\frac{p}{p+1}} \right]$$

and the proof is concluded. □

Chapter 8

Quantitative Urysohn's type inequalities

In this Chapter we apply the results of the previous chapter (namely Theorem 6.1.3 and Theorem 6.1.2) to derive quantitative versions of some Brunn-Minkowski and Urysohn type inequalities for functionals that can be written in terms of the solutions of suitable elliptic boundary value problems.

In Section 8.1 we consider the particular case of the torsional rigidity as a toy model for which we carry out all the computations; in Section 8.2 we present the same kind of quantitative results for a wide class of elliptic operators.

Throughtout the chapter, given an open bounded convex set of $\mathbb{R}^n$, we denote by $\Omega^\sharp$ the ball such that

$$w(\Omega) = w(\Omega^\sharp)$$

where $w(\cdot)$ is the mean-with defined in (6.14) in Section 6.2.

8.1 A toy model: the torsional rigidity

Let us recall the definition of the torsional rigidity $\tau(\Omega)$ of a bounded convex set (with non empty interior) Ω:

$$\frac{1}{\tau(\Omega)} = \inf\Big\{\frac{\int_\Omega |Du|^2\,dx}{(\int_\Omega |u|\,dx)^2} \;:\; u \in W_0^{1,2}(\mathrm{int}(\Omega)),\ \int_\Omega |u|\,dx < 0\Big\}. \tag{8.1}$$

Take u the unique solution of

$$\begin{cases} \Delta u = -2 & \text{in } \mathrm{int}(\Omega) \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{8.2}$$

Then we have

$$\tau(\Omega) = \int_\Omega u\,dx. \tag{8.3}$$

We recall an useful geometric property satisfied by the solutions of problem (8.2) (see [123] and [118] for details):

Proposition 8.1.1. *If u is the solution to problem* (8.2) *then u is $\frac{1}{2}$-concave, i.e. the function*

$$v(x) = \sqrt{u(x)}$$

is concave in Ω.

Finally we recall a comparison result for solutions of problem (8.2) in different domains (see [59] and [145] for details):

Proposition 8.1.2. *Let Ω_0, Ω_1 be open bounded convex sets (with nonempy interior) $\lambda \in [0,1]$ and $\Omega_\lambda = (1-\lambda)\Omega_0 + \lambda\Omega_1$, Let u_i be the solution of problem* (8.2) *in K_i, $i = 0,1,\lambda$. Then*

$$u_\lambda((1-\lambda)x+\lambda y)^{\frac{1}{2}} \geq (1-\lambda)u_0(x)^{\frac{1}{2}} + \lambda u_1(y)^{\frac{1}{2}} \quad \forall x \in \Omega_0,\, y \in \Omega_1.$$

The torsional rigidity satisfies the following Brunn-Minkowski inequality, whose essentially stems from (8.3) and the above propositions.

Proposition 8.1.3. *Under the assumptions and notations of Proposition 8.1.2, we have*

$$\tau(1-\lambda)\Omega_0 + \lambda\,\Omega_1) \geq \mathscr{M}_{\frac{1}{n+2}}\left(\tau(\Omega_0), \tau(\Omega_1), \lambda\right), \tag{8.4}$$

Our first main result is the following refinement of (8.4).

Theorem 8.1.4. *Under the assumptions and notations of Proposition 8.1.2, we have then the following strengthened versions of* (8.4)*:*

$$\tau(\Omega_\lambda) \geq \mathscr{M}_{\frac{1}{n+2}}\left(\tau(\Omega_0), \tau(\Omega_1), \lambda\right) + \beta\, H_0(\Omega_0, \Omega_1)^{3(n+1)}, \tag{8.5}$$

$$\tau(\Omega_\lambda) \geq \mathscr{M}_{\frac{1}{n+2}}\left(\tau(\Omega_0), \tau(\Omega_1), \lambda\right) + \delta A(\Omega_0, \Omega_1)^6, \tag{8.6}$$

where β and δ are as in Remark 6.1.6 of Section 6.1 with $p = 1/2$ (and $I_i = \tau(\Omega_i)$ for $i = 0,1$).

Proof. Thanks to Proposition 8.1.1 and Proposition 8.1.2, it is possible to apply Theorem 6.1.2 and Theorem 6.1.3 with $p = 1/2$ and $h = u_\lambda$. Then it is easily seen that (6.7) and (6.9) precisely reads as (8.5) and (8.6) respectively, thanks to (8.3). $\square$

In the following proposition we state an Urysohn type inequality for torsional rigidity, which can be retrieved from more general results in [49, 145] and it was already sketched in [40]. For a better understanding of our results, we will an explicit proof in the following paragraph.

Proposition 8.1.5. *Let Ω be an open bounded convex set in $\mathbb{R}^n$ and let $\Omega^\sharp$ be the ball with the same mean-width of Ω. Then it holds*

$$\tau(\Omega) \leq \tau(\Omega^\sharp) \tag{8.7}$$

and equality holds if and only if $\Omega = \Omega^\sharp$.

The content of the next theorem, which we will prove in the following paragraphs, amounts to two quantitative versions of (8.7), one in terms of the Hausdorff distance of Ω from $\Omega^\sharp$ and another one in terms of the relative asymmetry of Ω, as applications respectively of Theorem 6.1.2 and Theorem 6.1.3.

Theorem 8.1.6. *Let Ω be an open bounded convex subset of $\mathbb{R}^n, n \geq 2$ with centroid in the origin. Let $\Omega^\sharp$ be the ball with the same mean-width of Ω with center in the origin. Then the following hold*

$$\tau(\Omega^\sharp) \geq \tau(\Omega)\left(1+\mu H^{3(n+1)}\right), \tag{8.8}$$

$$\tau(\Omega^\sharp) \geq \tau(\Omega)\left(1+\nu A^{6}\right), \tag{8.9}$$

where $H = H(\Omega, \Omega^\sharp)$ and $A = \max\{A(\Omega, \Omega_\rho) : \rho$ any rotation in $\mathbb{R}^n\}$ are small enough, μ and ν are constants, the former depending on n, $\tau(\Omega)$ and the diameter of Ω, the latter depending only on n and $\tau(\Omega)$.

For explicit expressions (but not the optimal values) of the constants μ and ν involved in the previous theorem, see (8.14) and (8.16).

Remark 8.1.7. Theorem 8.1.4 and Theorem 8.1.6 are written as quantitative forms of the involved inequalities, but they can be obviously interpreted also as stability results for the same inequalities. For instance, (8.5) can be written so to show explicitly that $H_0(\Omega_0, \Omega_1)$ is small when we are close to equality in (8.4).

Urysohn inequality for torsional rigidity

We prove Proposition 8.1.5.

Proof. Since τ is invariant under translations, we can translate Ω in a way that the point of Steiner s of Ω coincides with the origin. We remind that the point of Steiner $s(\Omega)$ of a convex

set Ω is defined as
$$s(\Omega) = \frac{1}{\omega_n} \int_{S^{n-1}} \theta h(\Omega, \theta)\, d\mathscr{H}^{n-1}(\theta).$$
Using Hadwiger's Theorem (see [147]) there exists a sequence of rotations $\{\rho_k\}$ such that

$$\Omega_k = \frac{1}{k}(\rho_1 \Omega + \cdots + \rho_k \Omega) \tag{8.10}$$

converges, in the Hausdorff metric, to a ball.

We notice that Ω_k converges to $\Omega^\sharp$: in fact, since the mean-width is invariant under rigid motions and is additive under the Minkowski sum (see [147]), we get

$$w(\Omega_k) = w(\Omega) = b$$

for all k and so
$$w(\Omega^\sharp) = w(\Omega) = b.$$
Moreover $s(\Omega_k) = 0$ for all k for the same reason, and then $\Omega^\sharp$ is the ball with radius $r = \frac{b}{2}$ centered at 0.

Using (8.4) we get
$$\tau(\Omega_k) \geq \tau(\Omega) \quad \text{for all } k > 0, \tag{8.11}$$
since $\tau(\rho\Omega) = \tau(\Omega)$ for any rotation ρ.

Since Ω_k converges to $\Omega^\sharp$ in the Hausdorff metric when k goes to infinity, for every $m > 0$ there exists k_m such that
$$\Omega_k \subseteq B(0, r + \frac{1}{m})$$
for all $k \geq k_m$. Then
$$\tau(\Omega_{k_m}) \leq \tau(B(0, r + \frac{1}{m})). \tag{8.12}$$
By letting $m \to +\infty$, we finally get (8.7).

Regarding the equality case, obviously if Ω is a ball we get the equality in (8.7); conversely, the above proof gives

$$\tau(\Omega) \leq \tau(\Omega_k) \leq \tau(\Omega^\sharp) \quad \text{for all } k > 0,$$

then if equality holds in (8.7), we have

$$\tau(\Omega) = \tau(\Omega_k) = \tau(\Omega^\sharp)$$

for all $k > 0$ and thanks to the equality case in (8.4), we can conclude that Ω is a ball. $\square$

Quantitative Urysohn inequality

In this section we prove Theorem 8.1.6. Let us prove only (8.8); then (8.9) can be proved in the same way, using (8.6) in place of (8.5).

Proof. Let Ω_ρ be a rotation of Ω with center in the centroid of Ω and set

$$\tilde{\Omega} = \frac{1}{2}\Omega + \frac{1}{2}\Omega_\rho.$$

First notice that, since

$$w(\tilde{\Omega}) = w(\Omega),$$

by (8.7) we get

$$\tau(\tilde{\Omega}) \le \tau(\Omega^\sharp).$$

Since $\tau(\Omega_\rho) = \tau(\Omega)$, (8.5) gives

$$\tau(\tilde{\Omega}) \ge \tau(\Omega) + \beta' H_0(\Omega, \Omega_\rho)^{3(n+1)} \tag{8.13}$$

where

$$\beta' = \frac{|\Omega|^{3(n+1)/n}}{8(n+\tau(\Omega)^{-1})^3}\,[4\gamma_n d(\Omega)]^{-3(n+1)}$$

and $d(\Omega)$ is the diameter of Ω.

Since

$$H_0(\Omega, \Omega_\rho) = \frac{H(\Omega, \Omega_\rho)}{|\Omega|^{1/n}},$$

(8.13) becomes

$$\tau(\tilde{\Omega}) \ge \tau(\Omega)\left(1 + \mu H(\Omega, \Omega_\rho)^{3(n+1)}\right),$$

where

$$\mu = \tau(\Omega)^2\left[2^{2n+3}\gamma_n^{n+1} d(\Omega)^{n+1}(n\tau(\Omega)+1)\right]^{-3}. \tag{8.14}$$

Then we have just to show that we can find a rotation Ω_{ρ_0} of Ω such that

$$H(\Omega, \Omega_{\rho_0}) \ge H(\Omega, \Omega^\sharp)$$

We denote by h_Ω and $h_{\Omega^\sharp}$ the support functions of Ω and $\Omega^\sharp$ respectively. By definition of the support function (6.2) given in Section 6.2, we have

$$H(\Omega, \Omega^\sharp) = \max_{\theta \in S^{n-1}} |h_\Omega(\theta) - h_{\Omega^\sharp}(\theta)| = \max_{\theta \in S^{n-1}} |h_\Omega(\theta) - r| \tag{8.15}$$

where r is the radius of $\Omega^\sharp$, that is

$$r = \frac{w(\Omega)}{2} = \frac{1}{n\omega_n}\int_{S^{n-1}} h_\Omega(\theta)\,d\theta\,.$$

By the mean value Theorem and the continuity of h_Ω, there exists θ_0 such that

$$h_\Omega(\theta_0) = r.$$

Take $\bar\theta$ such that the maximum in (8.15) is attained at $\overline{\theta}$ and let ρ_0 be a rotation with center in the centroid of Ω such that

$$h_{\Omega_{\rho_0}}(\bar\theta) = h_\Omega(\theta_0).$$

Then thanks to (8.15) we get

$$H(\Omega,\Omega_{\rho_0}) \geq |h_\Omega(\bar\theta) - h_{\Omega_{\rho_0}}(\bar\theta)| = H(\Omega,\Omega^\sharp),$$

and we conclude the proof. □

Remark 8.1.8. During the proof we find an explicit value for the constant μ. The same can be done for the constant ν; here it is:

$$\nu = \frac{(1-2^{-1/n})^9\,\tau(\Omega)^2}{181^2\,n^{39}\,(n\tau(\Omega)+1)^3}\,. \tag{8.16}$$

Remark 8.1.9. Let us denote by $\Omega^\star$ a ball with the same measure as Ω. Then we notice that (8.7) is weaker than the well known St Venant's inequality (see [137])

$$\tau(\Omega) \leq \tau(\Omega^\star), \tag{8.17}$$

since τ is increasing with respect to inclusion and

$$\Omega^\star \subseteq \Omega^\sharp$$

by the classical Urysohn's inequality between mean width and volume of convex sets. This is due to the fact that the Laplacian or other kind of operator written in divergence form works better under Schwarz symmetrization. Moreover, any quantitative version of (8.17) would imply immediately the same quantitative result for (8.7). However, to our knowledge, no quantitative version of (8.17) have been proved yet.

8.2 Quantitative p-Minkowski convolutions and mean width rearragements

Results like Theorem 8.1.4 and Theorem 8.1.6 can be obtained for other functionals related to different elliptic operators. In particular, we could for instance derive similar results for the p-Laplacian, for the 2-Hessian operator in $\mathbb{R}^3$ and for the extremal Pucci's operator $\mathscr{P}^-_{\Lambda_1,\Lambda_2}$; the corresponding Brunn-Minkowski inequalities, as well as the needed concavity and comparison results, similar to Proposition 8.1.1 and Proposition 8.1.2, can be explicitly found in or retrieved from [60], [144] and [39, 145], respectively. Similar results has been obtained in [96] for some Monge-Ampère functionals, whose Brunn-Minkowski inequalities can be found in [104, 143].

An interesting and quite general formulation of some of the applications cited above can be given through the so-called *mean-width rearrangements*, introduced by Paolo Salani in [145] and which we recall hereafter.

Consider two convex sets Ω_0, Ω_1, fix $\lambda \in (0,1)$ and as usual set $\Omega_\lambda = (1-\lambda)\Omega_0 + \lambda\Omega_1$. Let u_0, u_1 and u_λ be the solutions of the corresponding Dirichlet problem

$$(P_i) \qquad \begin{cases} F_i(x,u_i,Du_i,D^2u_i) = 0 & \text{in } \Omega_i\,, \\ u_i = 0 & \text{on } \partial\Omega_i\,, \qquad i = 0,1,\mu \\ u_i > 0 & \text{in } \Omega_i\,, \end{cases}$$

where $F_i : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \times \mathscr{S}_n \to \mathbb{R}$ is a continuous proper degenerate elliptic operators (here and throughout $\mathscr{S}_n$ denotes the space of $n \times n$ real symmetric matrices).

For any $p \geq 0$ and for every fixed $\theta \in \mathbb{R}^n$ we define $G^{(\theta)}_{i,p} : \Omega_i \times (0,+\infty) \times \mathscr{S}_n \to \mathbb{R}$ as follows:

$$\begin{cases} G^{(\theta)}_{i,0}(x,t,A) = F_i(x,e^t,e^t\theta,e^tA) & \\ & \text{for } i = 0,1,\lambda\,. \\ G^{(\theta)}_{i,p}(x,t,A) = F_i(x,t^{\frac{1}{p}},t^{\frac{1}{p}-1}\theta,t^{\frac{1}{p}-3}A) \ \text{ if } p > 0 & \end{cases} \tag{8.18}$$

We say that F_0, F_1, F_λ satisfy the assumption $(A_{\lambda,p})$ if, for every fixed $\theta \in \mathbb{R}^n$, the following holds:

$$\begin{aligned} G^{(\theta)}_{\lambda,p}\big((1-\lambda)x_0+\lambda x_1,\ (1-\lambda)t_0+\lambda t_1, (1-\lambda)A_0+\lambda A_1\big) \geq \\ \min\{G^{(\theta)}_{0,p}(x_0,t_0,A_0);\, G^{(\theta)}_{1,p}(x_1,t_1,A_1)\} \end{aligned}$$

for every $x_0 \in \Omega_0$, $x_1 \in \Omega_1$, $t_0, t_1 > 0$ and $A_0, A_1 \in \mathscr{S}_n$.

Remark 8.2.1. If $F_0 = F_1 = F_\lambda$, we are simply requiring the operator G_p^θ to be quasi-concave, i.e. with convex superlevel sets.

In [145] it is proved that, under suitable assumptions, the p-Minkowski convolution $u_{p,\lambda}$ of the solutions u_0 and u_1 of (P_0) and (P_1) is a subsolution of problem (P_λ); we recall the precise statement in the following proposition.

Proposition 8.2.2. *Let $\lambda \in (0,1)$, Ω_i an open bounded convex set and u_i a classical solution of (P_i) for $i = 0,1$. Assume that F_0, F_1, F_λ satisfy the assumption $(A_{\lambda,p})$ for some $p \in [0,1)$. If $p > 0$, assume furthermore that for $i = 0,1$ it holds*

$$\liminf_{y \to x} \frac{\partial u_i(y)}{\partial \nu} > 0 \tag{8.19}$$

for every $x \in \partial\Omega_i$, where ν is any inward direction of Ω_i at x. Then $u_{p,\lambda}$ is a viscosity subsolution of (P_λ).

Then, when a comparison principle holds, it is possible to estimate the solution u_λ of (P_λ) by means of $u_{p,\lambda}$ and then by means of u_0 and u_1.

Corollary 8.2.3. *In the same assumption of the previous proposition and if F_λ satisfies a comparison principle, then*

$$u_\lambda((1-\lambda)x_0 + \lambda x_1) \geq M_p(u_0(x_0), u_1(x_1); \lambda) \tag{8.20}$$

for every $x_0 \in \Omega_0$, $x_1 \in \Omega_1$.

By a combination of the previous result with the BBL inequality, we can finally compare the L^r norms of the involved functions for any $r \in (0,+\infty)$; this is Corollary 4.2 of [145], which we recall now.

Corollary 8.2.4. *With the same assumptions and notation of Corollary 8.2.3, we have*

$$||u_\lambda||_{L^r(\Omega_\lambda)} \geq \mathscr{M}_{\frac{pr}{np+r}}\left(||u_0||_{L^r(\Omega_0)}, ||u_1||_{L^r(\Omega_1)}, \lambda\right) \quad \textit{for every } r \in (0,+\infty)\,. \tag{8.21}$$

Then it is probably clear as, by applying Theorem 6.1.2 and Theorem 6.1.3, we can easily get the refinements of (8.21) which are the content of the following theorem.

Theorem 8.2.5. *With the same assumptions and notations of Corollary 8.2.3, assume furthermore that for* $p > 0$

$$u_0 \text{ and } u_1 \text{ are } p\text{-concave functions} \tag{8.22}$$

(with convex compact supports Ω_0 *and* Ω_1 *respectively). Then, if* $H_0(\Omega_0, \Omega_1)$ *and* $A(\Omega_0, \Omega_1)$ *are small enough, it holds for every* $r \in (0, +\infty)$

$$||u_\lambda||^r_{L^r(\Omega_\lambda)} \geq \mathscr{M}_{\frac{p}{np+r}}\left(||u_0||_{L^r(\Omega_0)}, ||u_1||_{L^r(\Omega_1)}, \lambda\right) + \beta\, H_0(\Omega_0, \Omega_1)^{\frac{(n+1)(p+r)}{p}} \tag{8.23}$$

and

$$||u_\lambda||^r_{L^r(\Omega_\lambda)} \geq \mathscr{M}_{\frac{p}{np+r}}\left(||u_0||_{L^r(\Omega_0)}, ||u_1||_{L^r(\Omega_1)}, \lambda\right) + \delta\, A(\Omega_0, \Omega_1)^{\frac{2(p+r)}{p}}, \tag{8.24}$$

where δ, β *are constants depending only on* n, λ, p, $||u_0||^r_{L^r(\Omega_0)}$, $||u_1||^r_{L^r(\Omega_0)}$ *and on the measures of* Ω_0 *and* Ω_1.

Remark 8.2.6. When the operators F_0, F_1 satisfy suitable assumptions (see for example [137]), the p-concavity of the solutions u_0 and u_1 is known to hold, then there is no need of assuming (8.22). A particular case is when $F_0 = F_1 = F_\lambda$ and G_p^θ (defined in (8.18)) is quasi-concave (see Remark 8.2.1). In the following corollary we provide an example of Theorem 8.2.5 when the F_i satisfies these conditions.

Corollary 8.2.7. *Let* f *be a smooth nonnegative function defined in* $\mathbb{R}^n$. *Let* $\lambda \in (0,1)$ *and* Ω_0 *and* Ω_1 *be convex subsets of* $\mathbb{R}^n$ *and denote by* u_0, u_1 *and* u_λ *the solutions of*

$$\begin{cases} \Delta u_i + f(x) = 0 & \text{in } \Omega_i \\ \\ u_i = 0 & \text{on } \partial\Omega_i \end{cases}$$

for $i = 0, 1, \lambda$ *respectively, where* $\Omega_\lambda = (1-\lambda)\Omega_0 + \lambda\Omega_1$, *as usual. Assume* f *is* β*-concave for some* $\beta \geq 1$, *that is* f^β *is concave. Then* (8.23) *and* (8.24) *hold with*

$$p = \frac{\beta}{1+2\beta}.$$

Remark 8.2.8. In case f is a positive constant ($\beta = +\infty$), the same conclusions follow with $p = 1/2$ and we find the results of Theorem 8.1.4.

Let's see now as the technique above can be applied to improve some Talenti-like results (for operators not in divergence form) from [145]. We need first to set some notations. Let u be the solution of the problem

$$\begin{cases} F(x;u;Du;D^2u)=0 & \text{in } \Omega \\ u=0 & \text{on } \partial\Omega \\ u>0 & \text{in } \Omega \end{cases} \tag{8.25}$$

where $F(x;t;\xi;A)$ is a continuous proper elliptic operator acting on $\mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \times \mathscr{S}_n$ and Ω is an open bounded convex subset of $\mathbb{R}^n$. Denote by $\Omega^\sharp$ the ball withthe same mean width of Ω and v the solution of

$$\begin{cases} F(x;v;Dv;D^2v)=0 & \text{in } \Omega^\sharp \\ v=0 & \text{on } \partial\Omega^\sharp \\ v>0 & \text{in } \Omega^\sharp \end{cases} \tag{8.26}$$

We consider F a rotationally invariant operator, i.e.

$$F(\rho x;u;\rho\theta;\rho A\rho^T) = F(x;u;\theta;A)$$

for every $(x;u;\theta;A) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \times \mathscr{S}_n$ and every rotation $\rho \in \mathrm{SO}(n)$.

Remark 8.2.9. Let $\rho \in SO(n)$ and denote by Ω_ρ a rotation of Ω and by $u_\rho(x) = u(\rho^{-1}x)$ for $x \in \Omega_\rho$ a rotation of u. We remark that we consider rotationally invariant operators since the proof of Proposition 8.2.10 relies mainly on the fact that if u is a solution of (8.25) in Ω, then u_ρ is a solution of (8.25) in Ω_ρ.

Notice that F is rotationally invariant when it depends on x; θ and A only in terms of $|x|, |\theta|$ and the eigenvalues of A, respectively.

Given the operator F, a real number $p > 0$ and a vector $\theta \in \mathbb{R}^n$, we set

$$G_p^{(\theta)}(x,t,A) = F(x, t^{\frac{1}{p}}, t^{\frac{1}{p}-1} t^{\frac{1}{p}-3} A) \quad (x,t,A) \in \mathbb{R}^N \times [0,\infty) \times \mathscr{S}_n.$$

Then we the following holds (see [145]).

Proposition 8.2.10. *Let Ω be a bounded open convex set in $\mathbb{R}^n$ and u a solution of* (8.25) *where F is a rotationally invariant proper elliptic operator and assume u satisfies* (8.19). *Let $p \in (0,1)$ and assume that*

$$\textit{the set } \{(x,t,A) \in [0,\infty) \times \mathscr{S}_n : G_p^{(\theta)}(x,t,A) \geq 0 \textit{ is convex} \tag{8.27}$$

for every fixed $\theta \in \mathbb{R}^n$. *Then*

$$||v||_{L^r(\Omega^\sharp)} \geq ||u||_{L^r(\Omega)} \quad \textit{for every } r \in (0, +\infty] \tag{8.28}$$

Remark 8.2.11. Notice that assumption (8.27) is satisfied if the function $G_p^{(\theta)}$ is quasi-concave for every $\theta \in \mathbb{R}^n$, hence if it is q-concave for some $q \in \mathbb{R}$.

The proof of the above proposition is based on the definition of the so-called *mean-width rearrangement*. Roughly speaking, we associate to u a symmetrand $u_p^\sharp$ defined in the ball $\Omega^\sharp$ having the same mean width of Ω and, under suitable assumptions on the operator F (stated in Proposition (8.2.10)), we have a pointwise comparison between $u_p^\sharp$ and the solution v in $\Omega^\sharp$, that is

$$u_p^\sharp \leq v \quad \text{in } \Omega^\sharp \tag{8.29}$$

which leads to conclude (8.28).

The precise definition of $u_p^\sharp$ is actually quite involved. Here we just say that $u_p^\sharp$ is not equidistributed with u, in contrast with Schwarz symmetrization; indeed the measure of the super level sets of $u_p^\sharp$ is greater than the measure of the corresponding super level sets of u.

By following the same argument used to prove Theorem 8.1.6 and in particular applying Theorem 8.2.5, we derive the following quantitative version of Proposition (8.2.10).

Theorem 8.2.12. *With the same assumptions and notations of Theorem 8.1.6 and Proposition 8.2.10, assume furthermore that*

$$u \textit{ is } p\textit{-concave} \tag{8.30}$$

Then for all $r \in (0, +\infty)$, *it holds*

$$||v||^r_{L^r(\Omega^\sharp)} \geq ||u||_{L^r(\Omega)} + \eta\, H^{\frac{(n+1)(p+r)}{p}}$$

and

$$||v||^r_{L^r(\Omega^\sharp)} \geq ||u||_{L^r(\Omega)} + \sigma A^{\frac{2(p+r)}{p}},$$

where $H = H(\Omega, \Omega^\sharp)$ *and* $A = \max\{A(\Omega, \Omega_\rho) : \rho \textit{ rotation in } \mathbb{R}^n\}$ *are small enough,* η *and* σ *are constants, the former depending on n,* $||u||^r_{L^r(\Omega)}$ *and the diameter of* Ω*, the latter depending only on n and* $||u||^r_{L^r(\Omega)}$.

Remark 8.2.13. Notice that for $p = \frac{1}{2}$ and $r = 1$, Theorem 8.2.12 coincides with the quantitative result for the torsional rigidity stated in Theorem 8.1.6.

We remark that Theorem 8.2.5 and Theorem 8.2.12 apply also to nonlinear operators not in divergence form, for example the q-Laplacian, the Finsler laplacian and the Pucci Extremal operators.

References

[1] O. Alvarez, M. Bardi, Viscosity solutions methods for singular perturbations in deterministic and stochastic control, SIAM J. Control Optim. 40 (2001/02), 1159-1188.

[2] O. Alvarez, M. Bardi, Singular perturbations of nonlinear degenerate parabolic PDEs: a general convergence result, Arch. Ration. Mech. Anal. 170 (2003), 17-61.

[3] O. Alvarez, M. Bardi, *Ergodicity, stabilization, and singular perturbations for Bellman-Isaacs equations*, Mem. Amer. Math. Soc. (2010), no. 960, vi+77 pp.

[4] O. Alvarez, M. Bardi, C. Marchi, Multiscale problems and homogenization for second-order Hamilton-Jacobi equations, J. Differential Equations 243 (2007) 349-387.

[5] O. Alvarez, A. Tourin, Viscosity solutions of nonlinear integro-differential equations. Annales de l'institut Henri Poincaré (C) Analyse non linéaire 13 (1996), no. 3, 293-317.

[6] D. Applebaum, Lévy Processes and Stochastic Calculus. Second edition. Cambridge Studies in Advanced Mathematics, 116. Cambridge University Press, Cambridge, 2009. xxx+460 pp.

[7] M. Avellaneda, D. Boyer-Olson, J. Busca, P. Friz, Application of large deviation methods to the pricing of index options in finance, C.R. Math. Acad. Sci. Paris (2003), 336, 263-266.

[8] M. Arisawa, Ergodic problem for the Hamilton-Jacobi-Bellman equation II. Ann. Inst. H. Poincaré Anal. Non Linéaire 15, (1998), no.1, 1-24.

[9] M. Arisawa, P.-L. Lions ,On ergodic stochastic control, Comm. Partial Differential Equations, 23 (1998), no. 11-12, 2187-2217.

[10] S. Artstein, B. Klartag, V. Milman, The Santaló point of a function, and a functional form of Santaló inequality, Mathematika 51 (2004), no. 1-2, 33-48 (2005).

[11] S. Balbinot, Valore critico per Hamiltoniane non coercive e applicazioni a problemi di omogeneizzazione, Master thesis, University of Padova, 2012.

[12] K. M. Ball, K. J. Böröczky, Stability of the Prékopa-Leindler inequality, Mathematika, 56 (2010), no. 2, 339-356.

[13] K. M. Ball, K. J. Böröczky, Stability of some versions of the Prékopa-Leindler inequality, Monatsh. Math. 163 (2011), no. 1, 1-14.

[14] M. Bardi, I. Capuzzo-Dolcetta, Optimal control and viscosity solutions of Hamilton-Jacobi-Bellman equations, Birkhäuser Boston, Inc., Boston, MA, 1997.

[15] M. Bardi, A. Cesaroni, L. Manca, Convergence by viscosity methods in multiscale financial models with stochastic volatility, Siam J. Financial Math. 1 (2010), no.1, 230-265.

[16] M. Bardi, A. Cesaroni, Optimal control with random parameters: a multiscale approach, Eur. J. Control 17 (2011), no. 1, 30-45.

[17] M. Bardi, A. Cesaroni, D. Ghilli, Large deviations for some fast stochastic volatility models by viscosity methods, Discrete Contin. Dyn. Syst. A. 35 (2015), n. 9, 3965-3988.

[18] M. Bardi, A. Cesaroni, D. Ghilli, Large deviations for Orstein-Uhlenbeck fast stochastic volatility models by viscosity methods, Preprint 2015.

[19] G. Barles, A weak bernstein method for fully nonlinear elliptic equations. Differential Integral equations 4, (1991), no. 2, 241-262.

[20] G. Barles. $C^{0,\alpha}$-regularity and estimates for solutions of elliptic and parabolic equations by the Ishii-Lions method. Gakuto International Series, Mathematical Sciences and Applications, 30 (2008), 33-47.

[21] G. Barles. A short proof of the $C^{0,\alpha}$-regularity of viscosity subsolutions for superquadratic viscous Hamilton-Jacobi equations and applications. Nonlinear Anal., 73 (2010), no.1, 31-47.

[22] G. Barles, Solutions de viscosité des équations de Hamilton-Jacobi, Mathématiques and Applications 17, Springer-Verlag.

[23] G. Barles, E. Chasseigne, A. Ciomaga, C. Imbert, Lipschitz regularity of solutions for mixed integro-differential equations, J. Differential Equations 252 (2012), no. 11, 6012-6060.

[24] G. Barles, E.Chasseigne, C. Georgeline, E. R. Jacobsen, On neumann type problems for non-local equations set in a half space, Trans. Amer. Math. Soc. 366 (2014), no. 9, 4873-4917.

[25] G. Barles, E. Chasseigne, C. Imbert, On the Dirichlet problem for second order elliptic integro-differential equations, Indiana Univ. Math. J. 57, (2008), no. 1, 213-246.

[26] G. Barles, C. Georgeline,E. Jakobsen, On Neumann and oblique derivatives boundary conditions for second-order elliptic integro-differential equations, Journal of Differential Equations, 256, (2014), no. 4, 1368–1394.

[27] G. Barles, C. Imbert, Second-order elliptic integro-differential equations: viscosity solutions' theory revisited. Ann. Ist. H. Poincaré Anal. Non Linéaire 25, (2008), no.3, 567-585.

[28] G. Barles, H. Ishii, and H. Mitake. A new PDE approach to the large time asymptotics of solutions of Hamilton-Jacobi equations. To appear in Bull. Math. Sci., 2013.

[29] G. Barles, O. Ley, S. Koike, E. Topp, Regularity results and large time behavior for integro-differential equations with coercive hamiltonians, Calc. Var. Partial Differential Equations 54 (2015), no. 1, 539-572.

[30] G. Barles, P.L. Lions, Remarques sur les problèmes de réflexion oblique. C. R. Acad. Sci.Paris Sér. I Math. 320 (1995), no. 1, 69-74.

[31] G. Barles, B. Perthame, Comparison principle for Dirichlet-type Hamilton-Jacobi equations and singular perturbations of degenerated elliptic equations. Appl. Math. Optim. 21 (1990), no.1, 21-44.

[32] G. Barles, P. Souganidis, Space-time periodic solutions and long-time behavior of solutions of quasilinear parabolic equations. SIAM J. Math. Anal. (electronic) 32, (2001), no.6, 1311-1323.

[33] G. Barles, E. Topp, Lipschitz Regularity for Censored Subdiffusive Integro-Differential Equations with Superfractional Gradient Terms. Preprint 2014, arXiv:1412.2621.

[34] G. Barles, E. Topp, Existence, uniqueness and asymptotic behaviour for nonlocal parabolic problems with dominating gradient terms. Preprint 2014, arXiv:1404.7804.

[35] R. Bass, M. Kassmann, Hölder continuity of harmonic functions with respect to operators of variable orders . Comm. Partial Differential Equations 30 (2005), no. 7-9, 1249-1259.

[36] A. Bensoussan, Perturbation Methods in Optimal Control, John Wiley and Sons, Montrouge, France, 1988.

[37] A. Bensoussan, L. Boccardo, F. Murat, Homogenization of elliptic equations with principal part not in divergence form and Hamiltonian with quadratic growth. Comm. Pure Appl. Math. 39, (1986), no.6, 769–805.

[38] A. Bensoussan, J.-L. Lions, G. Papanicolaou, Asymptotic Analysis for periodic Structures. North-Holland, Amsterdam, 1978

[39] M. Bianchini, P. Salani, Power concavity for solutions of nonlinear elliptic problems in convex domains, Geometric Properties for Parabolic and Elliptic PDEs, 35-48, Springer INdAM Ser., 2, Springer, Milan, 2013.

[40] C. Bianchini, P. Salani, Concavity properties for elliptic free boundary problems. Nonlinear Anal. 71 (2009), no. 10, 4461-4470.

[41] K. Bogdan, K. Burdzy, Z.-Q Chen, Censored Stable Processes. Prob. Theory and Rel. Fields 127, (2003), no. 1, 89-152.

[42] C. Borell, Convex set functions in d-space, Period. Math. Hungar. 6 (1975), no. 2, 111-136.

[43] C. Borell, Capacitary inequalities of the Brunn-Minkowski type. Math. Ann. 263 (1983), no. 2, 179-184

[44] C. Borell, Greenian potentials and concavity, Math. Ann. 272 (1985), no 1, 155-160.

[45] V. S. Borkar and V. Gaitsgory, Singular perturbations in ergodic control of diffusions, SIAM J. Control Optim., 46 (2007), no.5, 1562–1577

[46] H. J. Brascamp and E. H. Lieb, Some inequalities for Gaussian measures and the long-range order of one-dimensional plasma, Functional Integration and Its Applications, ed. by A. M. Arthurs, Clarendon Press, Oxford 1975.

[47] H. J. Brascamp and E. H. Lieb, On extensions of the Brunn-Minkowski and Prékopa-Leindler theorems, including inequalities for log concave functions, and with an application to the diffusion equation, J. Functional Anal. 22 (1976), no. 4, 366-389.

[48] D. Bucur and I. Fragalà, Lower bounds for the Prékopa-Leindler deficit by some distances modulo translations, J. Convex Anal. 21 (2014), no. 1, 289-305.

[49] D. Bucur, I. Fragalà, J. Lamboley, Optimal convex shapes for concave functionals. ESAIM Control Optim. Calc. Var. 18 (2012), no. 3, 693-711.

[50] P.S. Bullen, Handbook of means and their inequalities. Mathematics and its Applications 560, Kluwer Academic Publishers Group, Dordrecht, 2003 (revised from the 1988 original [P. S. Bullen, D. S. Mitrinović and P. M. Vasić, Means and their inequalities, Reidel, Dordrecht]).

[51] L. Caffarelli and L. Silvestre. An extension problem related to the fractional Laplacian. Comm. Partial Differential Equations 32 (2007), no. 7-9, 1245-1260.

[52] F. Camilli, A. Cesaroni, C. Marchi, Homogenization and vanishing viscosity in fully nonlinear elliptic equations: rate of convergence estimates. Adv. Nonlinear Stud. 11 (2011), no.2, 405-428.

[53] I. Capuzzo Dolcetta, F. Leoni, and A. Porretta. H¨older estimates for degenerate elliptic equations with coercive Hamiltonians. Trans. Amer. Math. Soc., 362 (2010), no.9, 4511-4536.

[54] P. Cardaliaguet. A note on the regularity of solutions of Hamilton-Jacobi equations with superlinear growth in the gradient variable. ESAIM Control Optim. Calc. Var., 15 (2009), no.2. 367-376.

[55] P. Cardaliaguet and L. Silvestre. Hölder continuity to Hamilton-Jacobi equations with superquadratic growth in the gradient and unbounded right-hand side. Comm. Partial Differential Equations, 37 (2012), 1668- 1688.

[56] M. Christ, Near equality in the two-dimensional Brunn-Minkowski inequality. Preprint 2012, arXiv:1206.1965v2.

[57] M. Christ, Near equality in the Brunn-Minkowski inequality. Preprint 2012, arXiv:1207.5062v1

[58] A. Ciomaga, On the strong maximum principle for second-order nonlinear parabolic integro-differential equations. Adv. Differential Equations 17 (2012), no. 7-8, 635671.

[59] A. Colesanti, Brunn-Minkowki inequalities for variational functionals and related problems, Adv. Math. 194 (2005), no.1, 105-140.

[60] A. Colesanti, P. Cuoghi, P. Salani, Brunn-Minkowski inequalities for two functionals involving the p-Laplace operator, Appl. Anal. 85 (2006), no. 1-3, 45-66.

[61] A. Colesanti and P. Salani, The Brunn-Minkowski inequality for p-capacity of convex bodies, Math. Ann. 327 (2003), no. 3, 459-479.

[62] R. Cont and P. Tankov, Financial modelling with jump processes, Chapman and Hall/CRC Financial Mathematics Series. Chapman and Hall/CRC, Boca Raton, FL, 2004. xvi+535.

[63] D. Cordero-Erasquin, B. Klarta, Interpolations, convexity and geometric inequalities. Geometric aspects of functional analysis, 151-168, Lecture Notes in Math. 2050, Springer, Heidelberg, 2012.

[64] J. Coville, Maximum principles, sliding techniques and applications to nonlocal equations Electronic Journal of Differential Equations, (2007), no. 68, 23pp.

[65] J. Coville, Remarks on the Strong Maximum Principle for Nonlocal Operators Electron. J. Differential Equations (2008) no. 66, 10 pp.

[66] A. Cutrì, F. Da Lio, Comparison and existence results for evolutive non-coercive first-order Hamilton-Jacobi equations. ESAIM Control Optim. Calc. Var. 13 (2007), no. 3, 484-502.

[67] M. Crandall, H. Ishii, P.-L.Lions, User's guide to viscosity solutions of second order partial differential equations 27 (1992), no.1, 1-67.

[68] F. Da Lio, Comparison results for quasilinear equations in annular domains and applications, Comm. Partial Differential Equations 27 (2002), no. 1-2, 283-323.

[69] F. Da Lio, O. Ley, Uniqueness results for second order Bellman-Isaacs equations under quadratic growth assumptions and applications, SIAM J. Control Optim., 45 (2006) no. 1, 74-106.

[70] G. Dal Maso, H. Frankowska, Value functions for Boltza problems with discontinuous lagrangian and Hamilton-Jacobi inequality, ESAIM Control Optim. Calc. Var. 5 (2000), 369-393 (electronic).

[71] A. Dembo, O. Zeitouni, Large deviations techniques and applications, Springer, New York, 1998.

[72] A. Dinghas, Über eine Klasse superadditiver Mengenfunktionale von Brunn-Minkowski- Lusternikschem Typus, Math. Z. 68 (1957), 111-125.

[73] S. Dubuc, Critères de convexité et inégalités intégrales. (French. English summary) Ann. Inst. Fourier (Grenoble) 27 (1977), no. 1, 135-165.

[74] P. Dupuis, K. Spiliopoulos, Large deviations for multiscale problems via weak convergence mets, Stohodch. Process. Appl. 122 (2012), no.4, 1947-1987.

[75] R. Eldan, B. Klartag, Dimensionality and the stability of the Brunn-Minkowski inequality, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) 13 (2014), no. 4, 975-1007.

[76] L. C. Evans, The perturbed test function method for viscosity solutions of nonlinear PDE, Proc. Roy. Soc. Edinburgh Sect. A 111 (1989), no. 3-4, 359-375.

[77] L. C. Evans, Partial Differential Equations, Graduate studies of mathematics, American mathematical society, Providence, RI (1998) vol.19.

[78] L. C. Evans, H. Ishii, A PDE approach to some asymptotic problems concerning random differential equations with small noise intensities. Ann. Inst. H. Poincaré Anal. Non Linéaire 2 (1985), no.1, 1-20.

[79] J. Feng, J.-P. Fouque, R. Kumar, Small time asymptotic for fast mean-reverting stochstic volatility models, Ann. Appl. Probab. 22 (2012), no. 4, 1541-1575.

[80] J. Feng, T. G. Kurtz, Large deviations for stochastic processes, American Mathematical Society, Providence, RI, 2006.

[81] J. Feng, M. Forde, J.-P. Fouque, Short-maturity asymptotics for a fast mean-reverting Heston stochastic volatility model, SIAM J. Financial Math. 1 (2010), no.1, 126-141.

[82] A. Figalli, D. Jerison, Quantitative stability for the Brunn-Minkowski inequality. Preprint 2014, arXiv:1502.06513v1

[83] A. Figalli, F. Maggi, A. Pratelli, A refined Brunn-Minkowski inequality for convex sets, Annales IHP 26 (2009), no.6, 2511-2519.

[84] A. Figalli, F. Maggi, A. Pratelli, A mass transportation approach to quantitative isoperimetric inequality, Invent. Math. 182 (2010), no. 1, 167-211.

[85] W.H. Fleming, H. M. Soner, Controlled Markos processes and viscosity solutions, Springer, New York (2006).

[86] J.-P. Fouque, G. Papanicolaou, K.R. Sircar, Derivatives in financial markets with stochastic volatility. Cambridge university press, Cambridge (2000).

[87] J.-P. Fouque, G. Papanicolaou, R. Sircar, K. Solna, Singular perturbations in option pricing, SIAM J. Appl. Math. 63 (2003), no. 5, 1648-1665.

[88] J.-P. Fouque, G. Papanicolaou, R. Sircar, K. Solna, Multiscale stochastic volatility asymptotics, Multiscale Model. Simul. 2 (2003), no. 1, 22-42.

[89] J.-P. Fouque, G. Papanicolaou, R. Sircar, K. Solna, Multiscale stochastic volatility for equity, interest rate, and credit derivatives. Cambridge University Press, Cambridge, 2011.

[90] M. Freidlin, Functional integration and partial differential equations. Annals of Mathematics Studies, vol. 109, Princeton University Press, Princeton, (1985).

[91] A Friedman, Partial differential equations of parabolic type, Prentice-Hall, Inc., Englewood Cliffs, N. J., (1954).

[92] M. Fukushima, Oshima Y., Takeda M. Dirichlet forms and symmetric Markov processes. de Grueter Studies in Mathematics 19.Walter de Gruyter and Co., Berlin, 2011. x+489 pp.

[93] R. J. Gardner, The Brunn-Minkowski inequality, Bull. Amer. Math. Soc. 39 (2002), no. 3, 355-405.

[94] R. J. Gardner, The Brunn-Minkowski inequality: A survey with proofs, available at http://www.ac.wwu.edu/ gardner.

[95] D. Ghilli, On Neumann type problems for nonlocal Hamilton-Jacobi equations with dominating gradient terms. Submitted. arXiv:1601.04308 .

[96] D. Ghilli, P. Salani, Stability of isoperimetric type inequalities for some Monge-Ampère functionals, Ann. Mat. Pura Appl. (4) 193 (2014), no.3, 643-661.

[97] D. Ghilli, P. Salani, Quantitative Borell-Brascamp-Lieb inequalities for compactly supported power concave functions (and some applications). Submitted to Journal of Convex Analysis, arxiv:1502.0281.

[98] D. Gilbarg, N. S. Trudinger, Elliptic Partial Differential Equations of Second Order.

[99] H. Groemer, On the Brunn-Minkowski theorem, Geom. Dedicata 27 (1988),no.3, 357-71.

[100] Q. Y. Guan, The integration by part of the regional fractional Laplacian. Commun. Math. Phys. 266 (2006), 289-329.

[101] Q. Y. Guan, Z.M. Ma, Reflected symmetric α-stable process and regional fractional Laplacian. Probab. Theory Relat. Fields 134 (2006), no.4, 649-694.

[102] H. Hadwiger, D. Ohmann, Brunn-Minkowskischer Satz und Isoperimetrie, Math. Zeit.66 (1956), 1-8. MR 18-595c.

[103] G.H. Hardy, J.E. Littlewood, G. Polya, Inequalities, 2d ed. Cambridge, at the University Press (1952).

[104] D. Hartenstine, Brunn-Minkowsky-type inequalities related to the Monge-Ampère equation, Advanced Nonlinear Studies 9 (2009), no.2, 277-294.

[105] R. Z. Hasminskii, Stochastic Stability of Differential Equations, Sijthoff and Noordhoff, Alphen aan denRijn, The Netherlands, Germantown, MD (1980).

[106] R. Henstock, A. M. Macbeath, On the measure of sum sets, I. The theorems of Brunn, Minkowski and Lusternik, Proc. London Math. Soc.(3) 3, (1953), 182-194.

[107] N. Ichihara, Recurrence and transience of optimal feedback processes associated with Bellman equations of ergodic type, SIAM J. Control Optim. 49 (2011), no. 5, 1938–1960.

[108] N. Ichihara, S. Sheu, Large time behavior of solutions of Hamilton-Jacobi-Bellman equations with quadratic nonlinearity in gradient, Siam J. Math. Anal. 45 (2013), no.1, 279–306.

[109] K. Ishige, P. Salani, Parabolic power concavity and parabolic boundary value problems, Math. Ann. 358 (2014), no 3-4, 1091-1117.

[110] H. Ishii, Perron's method for hamilton-jacobi equations. Duke Math. J. 55, (1987), no.3,369-384.

[111] H. Ishii, S. Koike. Viscosity solutions for monotone systems of second-order elliptic PDEs. Comm. Partial Differential Equations, 16 (1991), no. 6-7, 1095-1128.

[112] H. Ishii, P.-L. Lions. Viscosity solutions of fully nonlinear second-order elliptic partial differential equations. J. Differential Equations 83 (1990), no.1, 26-78.

[113] N. Jacob, Pseudo differential operators and Markov process. Vol III. Markov process and Applications. Imperial College Press, London, 2005. xxviii+474 pp.

[114] R. Jensen, P.-L. Lions, Some asymptotic problems in fully nonlinear elliptic equations and stochastic control. Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 11, (1984) 129–176.

[115] Y. Kabanov, S. Pergamenshchikov, Two-scale stochastic systems. Asymptotic analysis and control, Springer-Verlag, Berlin (2003).

[116] H. Kaise, S. Sheu, On the structure of solution of ergodic type Bellman equation related to risk-sensitive control, Ann. Probab. 34, (2006), no.1, 284-320.

[117] B. Kawohl, Geometrical properties of level sets of solutions to elliptic problems. Nonlinear functional analysis and its applications, Part 2 (Berkeley, Calif. 1983) 25-36 Proc. Sympos. Pure Math., 45, Part 2, Amer. Math. Soc., Providence, RI, 1986.

[118] A.U. Kennington, Power concavity and boundary value problems , Indiana University Math. J. 34 (1985), no. 3, 687-704.

[119] Kim P., Weak convergence of censored and reflected stable processes. Stochastic Process. Appl. 116 (2006), no. 12, 1792-1814

[120] B. Klartag, Marginals of geometric inequalities. Geometric aspects of functional analysis,133-166, Lecture Notes in Math. 1910, Springer, Berlin (2007).

[121] H. J. Kushner, Weak Convergence Methods and Singularly Perturbed Stochastic Control and Filtering Problems, Birkhäuser, Boston (1990).

[122] H.J. Kushner ,Large deviations for two-time-scale diffusions, with delays, Appl. Math. Optim. 62 (2010), no. 3, 295-322.

[123] L.G. Makar-Limanov, The solution of the Dirichlet problem for the equation $\Delta u = -1$ in a convex region, Mat. Zametki 9 (1971) 89-92 (Russian). English translation in Math. Notes 9 (1971) 52-53 .

[124] L. Leindler, On a certain converse of Hölder's inequality, II . Acta Sci. Math. (Szeged) 33 (1972), no.3-4, 217-223. (English transl. of Dokl. Akad. Nauk. SSSR 165 (1965) 33-36).

[125] O. Ley, V. Duc Nguyen, Gradient bounds for nonlinear degenerate parabolic equations and application to large time behavior of systems, Nonlinear Analysis: Theory, Methods and Applications, ELsevier, 130 76-101 (2016).

[126] P.-L. Lions. Generalized solutions of Hamilton-Jacobi equations. Pitman (Advanced Publishing Program), Boston, Mass.(1982).

[127] P.-L. Lions, M. Musiela, Ergodicity of Diffusion Processes, manuscript (2002).

[128] P.-L. Lions, G. Papanicolau, S. R. S. Varadhan, Homogeneization of Hamilton-Jacobi equations, unpublished (1986).

[129] P. -L. Lions, P. E. Souganidis, Homogenization of degenerate second-order PDE in periodic and almost periodic environments and applications, Ann. I. H. Poincaré-AN 22 (2005), 667-677.

[130] P.-L. Lions, A.-S. Sznitman, Stochastic differential equations with reflecting boundary conditions. Comm. Pure Appl. Math. 37 (1984), no. 4, 511-537.

[131] R. Lipster, Large deviations for two scaled diffusions, Probab. Theory Relat. Fields 106, (1996), no.1, 71-104.

[132] L. Lorenzi, M. Bertoldi, Analytical Methods for Markov Semigroups,Pure and Applied Mathematics (Boca Raton), 283. Chapman and Hall/CRC, Boca Raton, FL, 2007. xxxii+526 pp.

[133] L. A. Lusternik, Die Brunn-Minkowskische Ungleichung fur beliebige messbare Mengen, C. R. (Doklady) Acad. Sci. URSS 8 (1935), 55-58.

[134] E. Pardoux, A. Yu. Veretennikov, On the Poisson equation and diffusion approximation, I, Ann. Probab. 29 (2001), no.3, 1061–1085.

[135] E. Pardoux, A. Yu. Veretennikov, On the Poisson equation and diffusion approximation, II, Ann. Probab. 31 (2003), no.3, 1166–1192.

[136] E. Pardoux, A. Yu. Veretennikov, On the Poisson equation and diffusion approximation, III, Ann. Probab. 33 (2005), no.3, 1111–1133

[137] G. Polya, G. Szëgö, Isoperimetric Inequalities in Mathematical Physics. Annals of Mathematical Studies, no. 27, Princeton University Press, Princeton, N. J. 1951. xvi+279 pp.

[138] A. Prékopa, Logarithmic concave measures with application to stochastic programming. Acta Sci. Math. (Szeged) 32 (1971) 301-316.

[139] R. T. Rockafellar, Convex Analysis, Princeton Mathematical Series, No. 28, Princeton University Press, Princeton, NJ (1970).

[140] F. Rezakhanlou, J.E. Tarver, Homogenization for stochastic Hamilton-Jacobi equations.Arch. Rational Mech. Anal. 151 (2000), no.4, 277-309.

[141] M. V. Safonov, On the classical solution of nonlinear elliptic equations of second order, Izv. Akad. Nauk SSSR Ser. Mat. 52 (1988), no. 6, 1272–1287.

[142] Sakaguchi, S., Concavity properties of solutions to some degenerate quasilinear elliptic Dirichlet problems , Ann Scuola Norm Sup Pisa, Serie IV (Classe di Scienze) 14 (1987), 403-421.

[143] P. Salani, A Brunn-Minkowski inequality for the Monge-Ampère eigenvalue, Adv. Math. 194 (2005), no.1, 67-86.

[144] P. Salani, Convexity of solutions and Brunn-Minkowski inequalities for Hessian equations in R^3 , Adv. Math. 229 (2012), no. 3, 1924-1948.

[145] P. Salani, Combination and mean width rearrangements of solutions of elliptic equations in convex sets, Ann. Inst. H. Poincaré Anal. Non Linéaire 32 (2015), no. 4, 763–783.

[146] A. Sayah, Équations d'hamilton-jacobi du premier ordre avec termes intégro-différentiels. ii. existence de solutions de viscosité. Comm. Partial Differential Equations 16 (1991), no.6-7, 1075-1093.

[147] R. Schneider, Convex bodies, The Brunn-Minkowski theory, Encyclopedia of Mathematics and its Applications, 44, Cambridge University Press, Cambridge, 1993.

[148] K. Spiliopoulos, Large Deviations and Importance Sampling for Systems of Slow-Fast Motion, Appl Math Optim 67 (2013), no.1, 123-161.

[149] T. Strömberg, The Operation of Infimal Convolution. Dissertationes Mathematicae (Rozprawy Matematyczne) 352 (1996), 58 pp.

[150] A. Takahashi, K. Yamamoto, A Remark on a Singular Perturbation Method for Option Pricing under a Stochastic Volatility, Asia-Pacific Financial Markets, 16 (2009), 333-345.

[151] T. T. Tchamba, Large time behavior of solutions of viscous hamilton-jacobi equations with superquadratic hamiltonian. Asymptot.Anal. 66 (2019), 161-186.

[152] E. Topp, Existence and uniqueness for integro-differential equations with dominating drift terms, Comm. Partial Differential Equations 39 (2014), no. 8, 1523–1554.

[153] N.S. Trudinger, On regularity and existence of viscoity solutions of nonlinear second order, elliptic equations, Partial differential equations and the Calculus of variations: Essays in Honor of Ennio de Giorgi, Progress in Nonlinear Differential Equations and their Applications, Birkhaüser Boston Inc., (1989).

[154] A. Yu. Veretennikov, On large deviations for SDEs with small diffusion and averaging. Stochastic Process. Appl. 89 (2000), no. 1, 69-79.

[155] D. Williams, Probability with Martingales, Cambridge University Press (1991).